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THE

ELEMENTS

OF

STATICS, DYNAMICS,

AND

HYDROSTATICS,

WITH AN APPENDIX ON THE LAWS OF LIGHT, THE FORMATION
OF IMAGES BY LENSES, AND THE NATURE OF SOUND.

BY

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PREFACE.

The design of the present work is two-fold.

It is intended to supply the requisite assistance to Students preparing for the Matriculation or B. A. pass examinations in the University of London. It therefore embraces all the Subjects in Natural Philosophy prescribed by the Senate for these examinations, and the necessary portions may be read with such a knowledge of pure Mathematics, as in each instance the Candidate may be fairly supposed to possess.

The Author has also endeavoured to provide a First Book in Natural Philosophy, by the aid of which, this study may be commenced in the earliest part of a Mathematical Course. With the exception of a few sections, this work may be read by any one who has mastered the First Book of Euclid, and is acquainted with the properties of similar triangles, and with the notation and simpler processes of Arithmetical Algebra.

The Sections marked by an asterisk may be omitted by the Student preparing for Matriculation.

Western College, Plymouth, December, 1840.

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STATICS.

CHAPTER I.

DEFINITIONS AND AXIOMS.

1. WHATEVER can either produce motion in or affect the notion of, any body is termed *force.*

2. If any number of forces acting upon a body counterbalance each other, so that the body remains at rest, those forces are said to be *in equilibrium.*

3. *Statics* is the science that investigates the relations which must subsist between the forces acting upon a body, in order that they may be in equilibrium.

4. The *single force,* if such there be, that represents the combined effect of several forces acting upon a body, is termed the *resultant* of those forces. With respect to the resultant, these several forces are termed *components,* or *component forces.*

Composition of forces takes place when two or more forces are replaced by a single force equivalent to them.

Resolution of forces takes place when a single force is replaced by two or more forces equivalent to it.

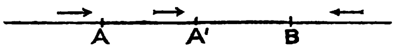
5. A ***rigid*** **body is one the relative position of whose particles is supposed to be invariable.**

6. *Axioms*, i.—If two forces acting upon a rigid body in exactly opposite directions are in equilibrium, they are equal.

ii.—If two equal forces act upon a rigid body in opposite directions, they will be in equilibrium.

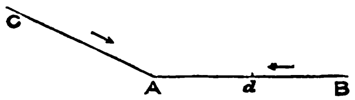
iii.—The effect of any forces acting upon a rigid body is unaltered by the introduction or the removal of any number of forces that are mutually in equilibrium.

7. *Transmissibility of force.* A force acting upon a rigid body may be supposed to act at any point, within the body, in the line of its direction. Thus, if a force P act upon a rigid body at A and along the line AB in the direction of the arrow, we may suppose P to be applied at any other point A' in the line AB within the body. For at A', and B, introduce two equal and opposite forces of the same magnitude as P. At A and B we have two equal and opposite forces. These forces are (ax. ii.) in equilibrium, and may therefore by ax. iii. be removed. There will then remain only a force P at A' of the same magnitude as the original force, and acting in the same direction.



8. *If two forces be in equilibrium their directions must be exactly opposite.*

For, if possible, let two forces acting along the lines CA, BA, be in equilibrium; since there is equilibrium, there



will not less be equilibrium, if any point be supposed fixed. Let d in the line AB be such a point, the force acting along AB will be met by the re-action of the fixed point d. The force CA will then remain alone, and will turn the body round the fixed point d, and therefore there will not be equilibrium.—Consequently if two forces be in equilibrium, their directions must be exactly opposed, and hence by axiom i. they are also equal in magnitude.

9. Hence if any number of forces acting upon a point be in equilibrium, any one force must be equal in magnitude and opposite in direction to the resultant of the remaining forces.

The expression equal in magnitude and opposite in direction is for convenience sake shortened into "equal and opposite."

10. Force when transmitted by means of a cord is sometimes spoken of as the *tension* in the cord.

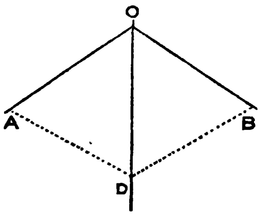
CHAPTER II.

THE COMPOSITION AND RESOLUTION OF FORCES ACTING UPON A POINT.

11. *Parallelogram of forces.* The proposition usually denoted by this term may be enunciated as follows: *If two forces acting upon a point are represented in magnitude and direction by the two sides of a parallelogram, then will their resultant also be represented in magnitude and direction, by the diagonal drawn through the given point.*

This proposition is proved by the following successive steps.

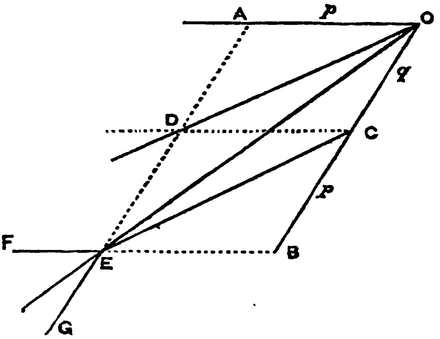
First: If two equal forces act upon a point the *direction* of the Resultant is that of the diagonal.



Let $A\,O$, $B\,O$ represent the magnitude and direction of two equal forces acting upon the point O, the diagonal $O\,D$

must be the direction of the resultant: For since $OADB$ is an equilateral parallelogram, the diagonal OD bisects the angle AOB; and since the forces acting upon O are equal, there can be no reason why the Resultant should pass nearer to the one than to the other; it must therefore pass at an equal distance from both: consequently it must pass along OD.

Secondly: If the direction of the resultant be that of the diagonal in the case of the two forces whose magnitudes are p and q respectively: it is so also, in the case of two forces whose magnitudes are p and $p + q$ respectively.



Let AO, BO represent two forces of the magnitudes p and $p + q$ respectively, acting upon the point O, and suppose them to be kept in equilibrium by some unknown force acting upon O. Make $BC = AO$: then CO will represent the magnitude of the force q.

Complete the parallelograms AC and DB. Draw the diagonals OD, OE and CE. For the forces p and q in AO and CO substitute their resultant, which by the hypothesis is a force acting along $O\,D$. Suppose this force to act at D, (Art. 7); then resolving it, we have a force p acting along the line $D\,C$, and a force q acting along the line $D\,E$. Let these two forces act at C and E respectively. We have then at the point C two equal forces of the magnitude p, acting along $C\,D$ and $C\,B$. Substitute for these their resultant; which, by the former part of this article, is a force acting along $C\,E$. Let this force act at E, and resolve it. We shall have then at E a force p acting in $E\,F$ and two forces p and q acting in $E\,G$. We have consequently, without disturbing the equilibrium, removed the original forces p and $p + q$ from O to E. The resultant therefore of the given forces must pass through E. It must also pass through O, and therefore must act along the diagonal $O\,E$.

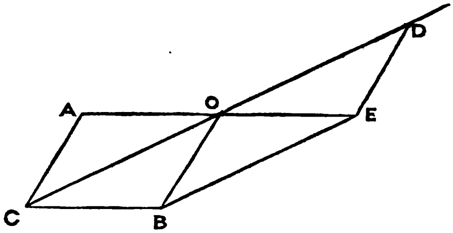
Thirdly: The direction of the Resultant is that of the Diagonal in the case of any two *commensurable forces.*

By the first step the proposition is true of forces p and p.
and ∴ by the second step it is true
of forces p and $p + p$ or p and $2\,p$
∴ .... p and $p + 2\,p$ or p and $3\,p$
and ∴ .... p and $p + 3\,p$ or p and $4\,p$
and so on.

It is true of $2\,p$ and p

∴ it is true of $2\,p$ and $2\,p + p$ or of $2\,p$ and $3\,p$
∴ ...... and $2\,p$ and $2\,p + 3\,p$ ... $2\,p$ and $5\,p$
and so on for any arithmetical ratio.

Fourthly. The diagonal represents the Resultant in magnitude.



Let $O\,A$, $O\,B$ represent two forces acting upon O. Complete the parallelogram $A\,O\,B\,C$. Draw the diagonal $O\,C$ and produce it through O. Make $O\,D$ equal to $O\,C$. Through D draw a line parallel to $O\,B$ cutting $A\,O$ produced in E. Join BE.

[Since $O\,D$ equals $O\,C$ and the angles $D\,O\,E$, $O\,D\,E$ are equal severally to $A\,O\,C$, $A\,C\,O$, the triangle $O\,D\,E$ equals the triangle $A\,C\,O$, and consequently the side $O\,E$ is equal to the side $A\,O$, and the side $D\,E$ to $A\,C$ and therefore to $O\,B$; and since $D\,E = O\,B$ and is also parallel to it, $O\,D\,E\,B$ is a parallelogram.]

The Resultant of $O\,A$ and $O\,B$ acts along $O\,C$: The forces in $O\,A$ and $O\,B$ will consequently be kept at rest by a force in $O\,D$ equal to this resultant. Suppose such a force to act in $O\,D$, then the point O will be kept at rest; but if three forces keep a point at rest, any one must be equal and opposite to the Resultant of the other two. Therefore $O\,A$ must be equal and opposite to the Resultant of $O\,B$ and the force in $O\,D$, Consequently $E\,O$ must be this Resultant.

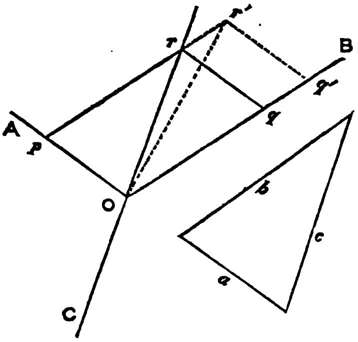
But the resultant of any two forces falls along the diagonal of the parallelogram whose sides represent these two forces in magnitude and direction. Therefore $O\,D$ must represent the magnitude of the force in $O\,D$, otherwise the resultant of it and $B\,O$ would not fall along $O\,E$. But the force in $O\,D$ was taken equal to the resultant of the forces in $A\,O$ and $B\,O$, and $O\,D = O\,C =$ the diagonal of the parallelogram $A\,B$. Therefore the resultant of any two forces is represented in magnitude as well as in direction, by the diagonal of the parallelogram, whose sides represent the components.

12. *Corol.* Hence for any force we may substitute two others, whose magnitude and direction are represented by the sides of any parallelogram, of which the line representing the given force forms the diagonal. And since an infinite number of parallograms can be drawn, having a given line for their diagonal, any force can be resolved into two others, in an infinite number of ways.

13. *If three forces* act upon a point and their resultant be required:—find the resultant of any two of them; the composition of this resultant, with the third force, will give the resultant of the three given forces. In like manner, the resultant of any number of forces acting upon a point, may be found.

14. *Triangle of Forces. If three forces acting upon a point be in equilibrium, and if any triangle be drawn, whose sides are severally parallel to their directions, the forces will be to one another as the sides of the triangle: and* conversely, *if the three forces be as the sides of the triangle they will be in equilibrium.*

Let P, Q and R be three forces in equilibrium acting upon the point O, and along the lines $O\,A$, $O\,B$ and $O\,C$ respectively.



In $C\,O$ produced take any point r; Draw $r\,p$ parallel to $O\,B$, and $r\,q$ parallel to $O\,A$.

Since the three forces are in equilibrium, the resultant of any two of these must be equal and opposite to the third force: Hence the resultant of P and Q must fall along $O\,r$.

Then if $O\,p$ be taken to represent the magnitude of the force P, $O\,q$ must represent the magnitude of the force Q: for if the magnitude of Q were represented by any other length $O\,q'$, then by the parallelogram of forces, the resultant of P and Q would fall along $O\,r'$. Then since the forces P and Q are represented in magnitude by the lines $O\,p$ and $O\,q$, their resultant is (Art. 10) represented in magnitude by $O\,r$. But the force R is equal to the resultant of the forces P and Q.

$$\therefore \quad P : Q : R :: Op : Oq : Or$$

But if any other triangle be drawn whose sides a, b, c, are severally parallel to Op, Oq, and Or, then (*Euclid* vi. *prop.* iv.)

$$a : b : c :: Op : Oq : Or$$

therefore

$$P : Q : R :: a : b : c.$$

Conversely: Let $P : Q : R :: a : b : c$. then shall the forces P, Q, and R be in equilibrium.

$$\text{For} \quad a : b : c :: Op : Oq : Or$$

$$\text{and} \therefore P : Q : R :: Op : Oq : Or$$

But if the forces P and Q are represented in magnitude and direction by the lines Op and Oq, then their resultant is represented by Or, and consequently is equal to R, but it is also opposite ; there is therefore equilibrium.

15. *Corol.* Since two triangles are similar, if the sides of the one are severally perpendicular to the sides of the other, * it follows from the preceding proposition, that if *three forces* acting upon a point be in equilibrium, and if any triangle be drawn whose sides are severally perpendicular to their directions, the forces will be to one another as the sides of this triangle.

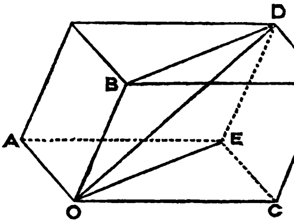
---

* Let the Student prove the following Geometrical proposition. The angle contained by the perpendiculars to two lines, is equal to the angle contained by the lines themselves.

This proposition will be frequently assumed in the following pages.

16. *If three forces, acting upon a point, be represented in magnitude and direction, by the three edges of a parallelopiped, their resultant will be represented in magnitude and direction by the diagonal.*

Let OA, OB, OC, the three edges of the parallelopiped BE represent in magnitude and direction three forces acting upon O.



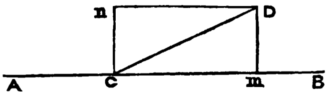
Since AC is a parallelogram, the diagonal OE represents the resultant of the forces OA and OC. Compound this resultant OE with the third force OB, then since BE is a parallelogram, their resultant is represented by OD; and therefore OD represents the resultant of the three forces OA, OB, OC.

17. *Lemma. No force can produce any effect in a direction perpendicular to its own.*

For there can be no reason why it should cause the body to move towards one side rather than towards the other; and consequently, it will not cause the body to move to either side.

18. *To determine the total effect of a force in any given direction.*

Resolve the force into two other forces, one perpendicular to, and the other acting along, the given direction. It follows from the preceding Lemma, that the last mentioned force is the thing sought.



Thus, let $C D$ represent a given force, and it be required to determine its total effect in the direction $A B$. Draw the rectangle $n m$. Then the force $C D$ may be resolved into two forces represented by $C n$ and $C m$. The force $C n$ produces no effect in the direction $A B$, and therefore $C m$ is the force required.

Let the given force be of the magnitude P, let $C D = r$ and $C m = x$, then,

$$\text{the required force} : P :: x : r$$

$$\therefore$$

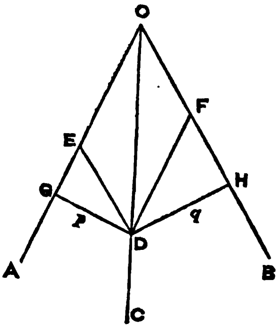
$$\text{The required force} = P\frac{x}{r}$$

19. Def. The product of a force, and the perpendicular distance of its direction from any given point, is termed the *moment* of the force about that point.

20. ***The moments of two forces about any point in the direction of their resultant are equal.***

Let P and Q be two forces acting respectively in the lines $A\ O$ and $B\ O$, and let $C\ O$ be the direction of their resultant. In $C\ O$ take any point D, and from D draw the perpendiculars $D\ G$ and $D\ H$. Let $D\ G = p$ and $D\ H = q$.

Then shall $P\,p = Q\,q$.



Complete the parallelogram $E\ F$. The triangles $D\ E\ G$, $D\ F\ H$ will be similar, and therefore

$$p : q :: DE : DF$$

$$:: DE : OE \quad \text{since } OE = DF$$

But by the triangle of forces

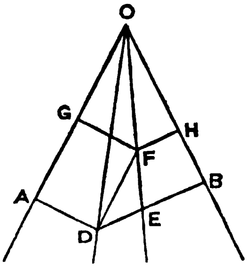
$$Q : P :: DE : OE$$

$$\therefore \quad Q : P :: p : q$$

$$\text{and} \therefore \quad P\,p = Q\,q.$$

21. *Conversely. If the moments of two forces about any point in their plane be equal, that point will lie in the direction of their Resultant.*

Let P and Q be two forces acting in $O\,A$ and $O\,B$ respectively, and let their moments about the point D be equal, that



is, let $P \times D\,A = Q \times D\,B$: then the resultant of the two forces shall fall along $O\,D$.

For, if possible, let the resultant not pass along $O\,D$, but along some other line $O\,E$. Through D draw $D\,F$ parallel to $O\,A$, and through F draw $F\,G$, $F\,H$ parallel to $D\,A$ and $D\,B$ respectively.

Then since F is a point in the resultant of P and Q, (Art. 20.)

$$P \times F\,G = Q \times F\,H$$

But $F\,G = A\,D$, for they are opposite sides of a rectangle

$\therefore \qquad P \times A\,D = Q \times F\,H$

but by hyp. $\quad P \times A\,D = Q \times D\,B$

and $\quad \therefore \qquad F\,H = D\,B$

but (Euclid vi. prop. iv.) . $E\,B : F\,H :: O\,B : O\,H$

and $O\,B$ is greater than $O\,H$; therefore $E\,B$ is greater than $F\,H$; and much more is $D\,B$ greater than $F\,H$. But it is absurd to suppose that $D\,B$ can be at the same time equal to, and greater, than $F H$. It is therefore also absurd to suppose that the resultant falls along $O E$. In like manner it can be shewn, that the resultant cannot fall along any other line than $O D$. Therefore the point D must lie in the resultant of the forces P and Q.

[The Student will perceive that the construction in Art. 20 and 21, fails, if the two forces be parallel; the proof therefore does not apply in that case. These propositions will be proved of parallel forces in the following chapter.]

EXAMPLES.

1. If P and Q be two forces acting upon a point at an angle of 60° then if R be their resultant,

$$R^2 = P^2 + Q^2 + P Q$$

2. If P and Q be two forces acting upon a point at an angle of 45°, then $R^2 = P^2 + Q^2 + \sqrt{2}. P Q$.

3. Let the angle between the directions of P and Q be 30°, then $R^2 = P^2 + Q^2 + \sqrt{3}. P Q$.

4. If a cord whose length is $2\,l$ be fastened at A and B lying in the same horizontal line, at a distance from each other, equal to $2\,a$, and if a smooth ring upon the cord sustain a weight W, then the tension in the cord

$$= \frac{W}{2\sqrt{1 - \frac{a^2}{l^2}}}.$$

5. If a weight of 20lb. slide along a perfectly smooth rod, inclined at an angle of 60° with the vertical line; what force is necessary to keep the weight at rest in any part of the rod?

6. In the previous question, if the rod were inclined at an angle of 45° shew that the force necessary to keep the weight at rest will be 14.142lbs.

7. Let *A*, *B* and *C* be three pegs in a vertical wall, over which passes a string with an equal weight attached at each end: determine the amount and direction of the pressure on each peg when *A B C* forms an isosceles triangle, whose base is horizontal, and whose vertical angle is 120°.

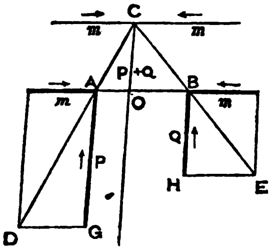
8. If the sides of any polygon represent severally the magnitudes and directions of a number of forces acting upon a point, then will the forces be in equilibrium.

CHAPTER III.

ON PARALLEL FORCES.

22. *To determine the magnitude and position of the resultant of two parallel forces* P *and* Q, *acting in the same direction.*

In the directions of P and Q take any points A and B, and introduce at A and B, two equal and opposite forces (m, m) The resultant of m and P is AD, and the resultant of m and Q is BE. Remove the points of application of these two resultants from A and B to C.



Resolve these into their original components, and we shall have two equal and opposite forces m, which will destroy each other; and two forces P and Q acting along $C\,O$ in the same direction. The resultant therefore is a force whose magnitude is $P + Q$ acting along $C\,O$.

By similarity of triangles

$$D\,G : G\,A :: A\,O : O\,C$$

$$\therefore \quad m : P :: A\,O : O\,C$$

$$\therefore P \times A\,O = m \times O\,C$$

and $E\,H : H\,B :: B\,O : O\,C$

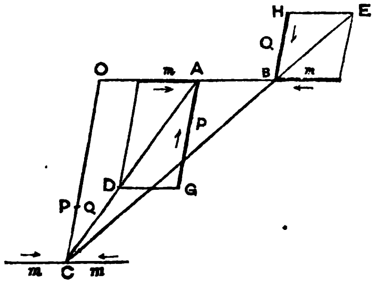
$$m : Q :: B\,O : O\,C$$

$$\therefore Q \times B\,O = m \times O\,C$$

$$\therefore \quad P \times A\,O = Q \times B\,O$$

23. *To find the resultant of two parallel forces P and Q, acting in different directions.*

As in the former case, take any two points A and B in the directions of P and Q, and introduce two equal and opposite forces $(m\,,\,m\,)$



Compound these with P and Q. Produce the directions of the resultants till they meet in C. Remove the point of application of these resultants from A and B to C. Resolve them into their original components, and we shall have two equal and opposite forces (m, m) acting in a line parallel to $A\,B$, which will destroy each other; and along $C\,O$ parallel to the directions of P and Q, we shall have two forces equal severally to P and Q, acting in different directions. The resultant then will be a force $P - Q$ acting along $C\,O$.

By similarity of triangles

$$m : P :: A\,O : C\,O \;\therefore\; m\,.\,C\,O = P \times A\,O$$

$$\text{and } m : Q :: B\,O : C\,O \;\therefore\; m\,.\,C\,O = Q \times B\,O$$

$$\therefore \qquad P \times A\,O = Q \times B\,O\,.$$

24. From these investigations we see, in the case of two parallel forces,

First:—That the resultant is always parallel to the components.

Secondly:—That when the components act in the same direction, the resultant is equal to their sum and passes between them; but when the components act in different directions, the resultant is equal to their difference, and passes outside the greater force.

Thirdly:—That if any line whatever (for the points A and B were taken quite arbitrarily) be drawn across the directions of the components, it will be cut by the resultant at a point such that the one force multiplied into its distance from the resultant measured along this line, equals the other force multiplied into its distance.

25. In Art. 23, if the forces P and Q had been equal, the construction would have been impossible. The lines AD, BE would have been parallel, and consequently the determination of the point C would have become an impossibility. The two forces will therefore have no resultant. Such a combination of forces is termed a couple.

26. *The resultant of any number of parallel forces acting in the same direction, equals the sum of the components.*

Let P_1, P_2, P_3, P_4 be the forces. Let R_1 be the resultant of P_1 and P_2, then

$$R_1 = P_1 + P_2$$

Let R_2 be the resultant of R_1 and P_3; then

$$\begin{aligned} R_2 &= R_1 + P_3 \\ &= P_1 + P_2 + P_3 \end{aligned}$$

The resultant of R_2 and P_4 will be the resultant of the given forces; let this be R then

$$\begin{aligned} R &= R_2 + P_4 \\ &= P_1 + P_2 + P_3 + P_4, \end{aligned}$$

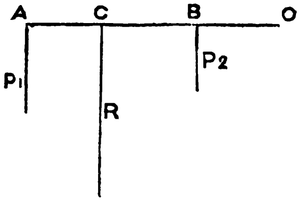
and similarly of any other number of parallel forces.

27. *The sum of the moments of any two parallel forces acting in the same direction, about any point in their plane lying without them, is equal to the moment of the resultant about that point.*

Let P_1 and P_2 be any two parallel forces acting in the same direction, and let R be their resultant; then if

p_1, p_2 and r be their distances respectively from the point O,

$$P_1\,p_1 + P_2\,p_2 = R\,r$$



For since R is the resultant of P_1 and P_2,

$$R = P_1 + P_2$$

and $$P_1 \times A\,C = P_2 \times B\,C$$

But $A\,C = p_1 - r$, and $B\,C = r - p_2$, therefore

$$P_1\,(p_1 - r) = P_2\,(r - p_2)$$

$$\therefore \quad P_1\,p_1 - P_1\,r = P_2\,r - P_2\,p_2$$

$$\therefore \quad P_1\,p_1 + P_2\,p_2 = (P_1 + P_2)\,r$$

$$= R\,r.$$

28. Hence :—*If any number of parallel forces act in the same plane and in the same direction, the sum of their moments about any point in the plane lying without them, is equal to the moment of the resultant about that point.*

Let the forces be P_1, P_2, P_3, P_4, &c., and their perpendicular distances from the given point p_1, p_2, p_3, p_4, &c., respectively.

Let R_1 be the resultant of P_1 and P_2, and its distance from the given point r_1 then by the preceding article,

$$R_1 \; r_1 = P_1 \, p_1 + P_2 \, p_2$$

Let R_2 be the resultant of R_1 and P_3, and its distance from the given point r_2.

$$\begin{aligned} \therefore \qquad R_2 \; r_2 &= R_1 \; r_1 + P_3 \; p_3 \\ &= P_1 \; p_1 + P_1 \; p_2 + P_3 \; p_3, \end{aligned}$$

and similarly for the rest, and hence if R be the resultant of all the forces, and its distance from the given point be r, then

$$R \, r, = P_1 \; p_1 + P_2 \; p_2 + P_3 \; p_3 + P_4 . \, p_4, + \ldots\ldots$$

29. *If any number of parallel forces act in the same plane, and in the same direction, and their moments be taken about any point in their resultant; then will the sum of the moments of the forces on one side of the resultant be equal to the sum of the moments of those on the other side.*

Let P_1, P_2, P_3, &c., be forces lying on one side of the resultant, and p_1, p_2, p_3, &c., their perpendicular distances from any given point in that resultant, and let Q_1, Q_2, Q_3, &c., denote the forces on the other side, and q_1, q_2, q_3, &c., their distances from the same point ... Then shall

$$\begin{aligned} &P_1, \, p_1, + P_2, \, p_2, + P_3, \, p_3, + \ldots \\ = \; &Q_1, \, q_1, + Q_2, \, q_2, + Q_3, \, q_3, + \ldots \end{aligned}$$

For let P be resultant of the forces P_1, P_2, &c., and let p be its distance from the given point, then since this point lies without these forces and in the same plane, it follows from Art. 28, that

$$P\,p = P_1, p_1, + P_2, p_2, + P_3, p_3, + \cdots$$

Similarly, if Q be the resultant of Q_1, Q_2, &c., and q its distance from the given point;

$$Q\,q = Q_1, q_1, + Q_2, q_2, + Q_3, q_3, + \cdots$$

but the resultant of the given forces is the resultant of P and Q, and $\therefore$ Art. 22.

$$P\,p = Q\,q$$

and $\therefore$

$$P_1, p_1, + P_2, p_2, + P_3, p_3, + \cdots$$
$$= Q_1\, q_1 + Q_2\, q_2 + Q_3\, q_3 + \cdots$$

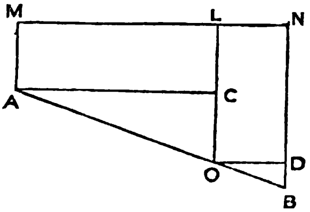
30. *Def.*—The product of a force, and the perpendicular distance of its point of application from any given plane, is termed the moment of the force with respect to the plane.

31. *If any two parallel forces act in the same direction, and their points of application lie both on the same side of a given plane; the sum of the moments of the forces with respect to the plane, equals the moment of the resultant.*

Let A and B be the points of application of the two parallel forces P_1 and P_2, and let perpendiculars drawn from A and B meet the given plane in M and N. Let the resultant pass through the point O, and let the perpendicular drawn from

O meet the plane in L. Let $A\,M = h_1$, $B\,N = h_2$, and $O\,L = h$. Then shall $R\,h = P_1\,h_1 + P_2\,h_2$.

For since R is the resultant of the two parallel forces P_1 and P_2 acting in the same direction,



$$R = P_1 + P_2$$

$$\therefore \quad R\,h = P_1\,h + P_2\,h$$

$$= P_1\,(h_1 + O\,C) + P_2\,(h_2 - B\,D)$$

$$= P_1\,h_1 + P_2\,h_2 + P_1 \times O\,C - P_2 \times B\,D$$

but since the resultant of P_1, and P_2 passes through O

$$P_1 : P_2 :: B\,O : A\,O$$

but by similarity of triangles

$$B\,D : O\,C :: B\,O : A\,O$$

$$\therefore \quad P_1 : P_2 :: B\,D : O\,C$$

$$\therefore \quad P_1 \times O\,C = P_2 \times B\,D$$

and $\therefore$

$$R\,h = P_1\,h_1 + P_2\,h_2.$$

32. If any number of parallel forces act in the same direction, and their points of application all lie on the same

side of a given plane, the sum of the moments of the forces with respect to the plane, is equal to the moment of the resultant.

Let the forces be P_1, P_2, P_3, &c., acting at points whose perpendicular distances from the plane are h_1, h_2, h_3, &c.

Let the resultant of P_1 and P_2 be R_1, let it act at a distance z_1 from the given plane, then by the preceding article;

$$R_1\ z_1 = P_1\ h_1 + P_2\ h_2.$$

Let the resultant of R_1 and P_3 be R_2, and let it act at a distance z_2 from the given plane, then

$$\begin{aligned} R_2\ z_2 &= R_1\ z_1 + P_3\ h_3 \\ &= P_1\ h_1 + P_2\ h_2 + P_3\ h_3, \end{aligned}$$

and in like manner for the rest. Hence if R be the resultant of all the forces, and h the distance of its point of application from the given plane.

$$R\,h = P_1\ h_1 + P_2\ h_2 + P_3\ h_3 + P_4\ h_4 + \dots$$

33. *To resolve a given force P into two parallel forces acting at given distances on each side of it.*

Let a and b be the given distances and let X and Y be the forces required, then P will be the resultant of X and Y, consequently (Art. 24.)

$$X + Y = P$$

and $$X a = Y b$$

$\therefore$ $$X + \frac{a}{b} X = P$$

$$X = P \frac{b}{a + b}.$$

Similarly $$Y = P \frac{a}{a + b}.$$

34. *To resolve a given force P into two parallel forces acting at given distances on the same side of it.*

Let a and b be the given distances, a being greater than b, and X and Y the required forces. From Art. 24 it appears that X and Y must act in opposite directions, and that the force nearer to P, viz. Y will act in the same direction as P,

hence $$Y - X = P$$

and $$b\,Y = a\,X$$

$\therefore$ $$\frac{a}{b} X - X = P$$

$\therefore$ $$X = P \frac{b}{a - b}$$

Similarly $$Y = P \frac{a}{a - b}.$$

35. *Ex.* A beam 10 ft. in length rests in a horizontal position upon two vertical props A and B, a weight of 30 lbs. is suspended at a distance of 7 ft. from the prop A, to find the pressure upon each of the props.

Let X = pressure upon A, and Y = pressure upon B, then since $P = 30$, $a = 7$, and $b = 3$, by Art. 33,

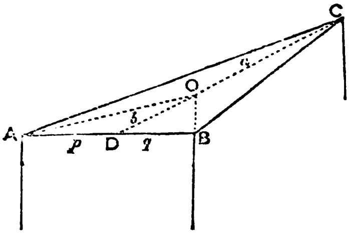
$$X = 30 \times \frac{3}{10} = 9 \text{ lbs.}$$

$$Y = 30 \times \frac{7}{10} = 21 \text{ lbs.}$$

*36. *Ex.* A weight W rests upon a triangular table to find the pressure upon each of the three legs.

Let the weight W rest upon the table at the point O. Join C, O, and let $C\,O$ produced cut the side $A\,B$ in D.

Let $C\,O = a$, $D\,O = b$, $A\,D = p$ and $D\,B = q$.



The weight W acts vertically, resolve it into two vertical forces at C and D, then by Art. 33 the resolved force at C is,

$$W \frac{b}{a + b} = \text{pressure upon the leg } C,$$

and the resolved force at D is $W \dfrac{a}{a + b}$. Let this be again resolved into two forces acting at A and B respectively, then

$$W \frac{a}{a + b} \cdot \frac{p}{p + q} = \text{pressure upon the leg } B.$$

and

$$W \frac{a}{a + b} \cdot \frac{q}{p + q} = \text{pressure upon the leg } A.$$

And hence if the table be divided into three triangles by joining O with each of the angular points, the pressure on either of the props will be to the whole weight, as the triangle opposite to it is to the area of the table :—for the preceding result gives

$$\frac{\text{pressure upon } C}{W} = \frac{b}{a+b}$$

but $AOD : ACD :: b : a+b$

and $BOD : BCD :: b : a+b$

$\therefore$ $AOD + BOD : ACD + BCD :: b : a+b$*

or $AOB : ABC :: b : a+b$

$\therefore$ $$\frac{\text{pressure upon } C}{W} = \frac{AOB}{ABC.}$$

Similarly

$$\frac{\text{pressure upon } B}{W} = \frac{a}{a+b} \cdot \frac{p}{p+q}$$
$$= \frac{AOC}{ACD} \times \frac{ACD}{ABC}$$
$$= \frac{AOC}{ABC}$$

and $$\frac{\text{pressure upon } A}{W} = \frac{a}{a+b} \cdot \frac{q}{p+q}$$
$$= \frac{BOC}{BCD} \times \frac{BCD}{ABC}$$
$$= \frac{BOC}{ABC}$$

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*Euclid B. v. Prop. xii. If the Student has not read the Fifth Book of Euclid, let him prove that if

$a : b :: p : q$

and $A : B :: p : q$

then $A + a : B + b :: p : q$

EXAMPLES.

1. If the parallel forces P and Q act in the same direction, and at the point A and B, and if $A\,B = a$ shew that the distance of the resultant from P, measured along the line $A\,B$, will be $\dfrac{a\,Q}{P + Q}$.

2. If P and Q act in opposite directions and P be the greater of the two forces, then the distance of the resultant from P will be $\dfrac{a\,Q}{P - Q}$.

3. If A, B and C be the points of application of three forces P_1, P_2, and P_3, acting in the same direction, shew that the resultant will pass between P_1 and P_2, if $P_1 \times A\,B$ be greater than $P_3 \times B\,C$, but between P_2 and P_3 if $P_1 \times A\,B$ be less than $P_3 \times B\,C$.

4. A beam $A\,B$, 10 feet in length, rests horizontally upon two vertical props A and B, and another beam $C\,D$, 20 feet in length, rests upon two vertical props C and D; a third beam, 30 feet in length, lies across the two former beams in such a way that the one extremity E is 3 feet from the prop A, and the other extremity F is 5 feet from the prop D: a weight of 60lbs. is attached to the third beam at a distance of 10 feet from F, shew that the pressures upon A, B, C, D, are 14, 6, 10, 30lbs. respectively.

5. A weight resting upon a horizontal table, exerts an equal pressure upon each of the three legs, determine the position of the weight.

CHAPTER IV.

ON THE CENTRE OF GRAVITY.

37. If any number of parallel forces act at given points in a rigid body, it can be shewn that there is a point through which the resultant will always pass, whatever the position in which the body may be placed.

This point is termed the centre of the parallel forces.

38. *Gravity* is the force which attracts every particle of matter towards the centre of the Earth. The *weight* of a body is the total force with which that body is drawn towards the Earth's centre: it is therefore the same as the resultant of the forces which, in consequence of the existence of gravity, act upon its several particles. These forces may, without sensible error, be regarded as parallel forces. The centre of these parallel forces is called the *centre of gravity.*

The centre of gravity of a body is then, the point through which in every position of the body will pass the resultant of all the forces which, in consequence of gravity, act upon its particles.

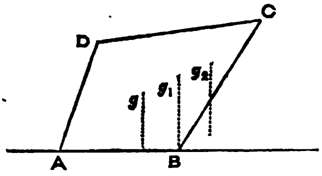
39. It follows from the definition, that if the centre of gravity of a body be fixed, the body will rest in any position. For in every position the resultant of all the forces arising from gravity passes through the fixed point, and being met by the resistance of that point, equilibrium is preserved.

40. Since the resultant may be always substituted in place of its component forces, it follows that in considering the influence of any weighty body in producing equilibrium, we may substitute a force equal to the weight of the body, and acting at its centre of gravity.

41. If a body suspended from any point be at rest, the centre of gravity must lie in the vertical line drawn through this point. For there are two forces in equilibrium, viz.: The weight of the body which acts vertically, and at the centre of gravity; and the reaction of the fixed point. By Art. 8 these forces must lie in the same straight line, and therefore the vertical line through the centre of gravity must pass through the fixed point.

Hence,—If a body be suspended successively from two different points, and the vertical lines passing through those points be drawn, the intersection of these two lines will be the centre of gravity.

42. A body placed with its base upon a plane surface, will stand or fall, according as the vertical line through its centre of gravity falls within or without the base.



Let *A B C D* be any body, whose base *A B* rests upon a plane surface. Let the centre of gravity be at *g*, so that the vertical line through g falls between $A\ B$. The weight of the body acting in this line is met by the resistance of the plane, and, consequently, will not cause the body to turn about either A or B.

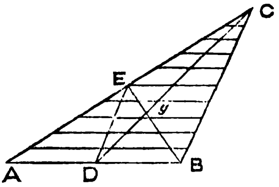
But if the centre of gravity be at g_2, so that the vertical line drawn through the centre of gravity falls without the base, the weight of the body, acting in this line, is not met by the re-action of the plane. There cannot, therefore, be equilibrium, but the body will turn over on to the side $B\ C$.

If the centre of gravity be at g_1, so that the vertical line drawn through it pass through one of the extremities of the base, the body will rest; for the weight acting in the vertical line is met by the resistance of the plane, a very slight disturbance will, however, cause the body to fall over.

43. The centre of any perfectly symmetrical figure must also be the centre of gravity, for there can be no reason why the centre of gravity should fall upon one side of that point more than upon another. Hence the centre of gravity of a line, is its point of bisection; of a parallelogram the intersection of its diagonals; of a parallelepiped the intersection of its diagonals; of a circle and sphere, their centres.

44. *To find the centre of gravity of a triangle.* Let $A\ B\ C$ be the given triangle. Bisect $A\ B$ in D and $A\ C$ in E. Draw $C\ D$, $B\ E$. If we suppose the triangle to be made up of an infinite number of lines parallel to $A\ B$; these lines will be bisected by $C\ D$. Consequently the centre of gravity of each of these lines will be in the line $C\ D$, and therefore

the centre of gravity of the triangle will lie in the line $C\,D$. For similar reasons the centre of gravity must lie in the line $B\,E$. Therefore g, the intersection of these two lines must be the centre of gravity required.



Join the points D, E. $D\,E$ will be parallel to $C\,B$, and the triangle $A\,D\,E$ similar to the triangle $A\,B\,C$. Hence,

$$A\,D : A\,B :: D\,E : B\,C,$$

but $$A\,D = \tfrac{1}{2}\,A\,B$$

$\therefore$ $$D\,E = \tfrac{1}{2}\,B\,C$$

Also the triangles $g\,D\,E$, $g\,B\,C$ are similar,

$\therefore$ $$g\,D : g\,C :: D\,E : B\,C.$$

$\therefore$ $$g\,D = \tfrac{1}{2}\,g\,C,$$

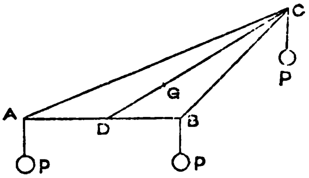
hence,

$$g\,D = \tfrac{1}{3}\,D\,C.$$

Hence the centre of gravity of a triangle lies in the line joining any vertex with the bisection of the opposite side, at a distance from the side of one third of this line.

45. *To find the centre of gravity of three equal weights suspended from the angular points of a triangular plate; the plate being supposed to be without weight.*

Let a weight P be suspended, from each of the angular points A, B, C of the triangular plate $A\,B\,C$. Since these weights hang vertically, they may be regarded as parallel forces, acting severally at the points of suspension. Bisect



$A\,B$ in D. Join D, C. Let $D\,G = \frac{1}{3}\,D\,C = \frac{1}{2}\,G\,C$. The resultant of the force P at A, and the force P at B, will be a force $2\,P$ acting at D. The resultant of $2\,P$ at D and P at C, will pass through G, for since $D\,G = \frac{1}{2}\,G\,C$,

$$2\,P \times D\,G = P \times G\,C,$$

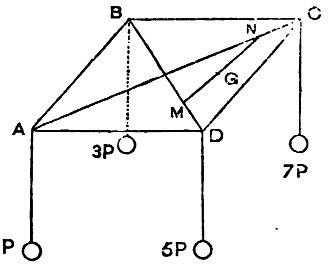
and, therefore, Art. 24, G is the point through which the resultant will pass, and it will pass through G, whatever the angle made by AB or $D\,C$ with the direction of the forces, (Art. 24, 3.) G is therefore the centre of gravity required.

By the preceding section G is the centre of gravity of the triangle $A\,B\,C$, whence it appears that the centre of gravity of a triangle corresponds with that of three equal forces applied at its angular points.

46. *To find the centre of gravity of four weights,* P, $3\,P$, $7\,P$, *and* $5\,P$, *suspended from the angular points of a square plate.*

Let $A\,B\,C\,D$ be the square plate, and let the weights P, $3\,P$, $7\,P$, and $5\,P$ be severally suspended from the points A, B, C, D.

Draw the diagonals, $A\,C$, $B\,D$. Take N so that $C\,N = \frac{1}{8}\,A\,C = \frac{1}{7}\,A\,N$, and M so that $D\,M = \frac{3}{8}\,B\,D = \frac{3}{5}\,B\,M$.



The resultant of P at A, and $7\,P$ at C is a force $8\,P$ at N, since $7\,P \times C\,N = P \times A\,N$. And the resultant of $3\,P$ at P and $5\,P$ at D is also a force $8\,P$ at M; since

$$5\,P \times D\,M = 3\,P \times B\,M.$$

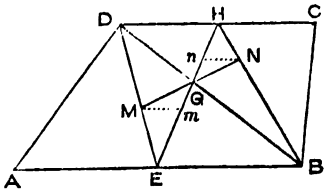
We have then in the place of the original forces two equal forces acting at M and N. Bisect $M\,N$ at G, then the resultant of these equal forces will pass through G, and G is the centre of gravity required.

*47. *To find the centre of gravity of a trapezium.*

Let $A\,B\,C\,D$ be the trapezium, in which the sides $A\,B$, $C\,D$ are parallel.

Bisect $A\,B$ in E, and $D\,C$ in H, and join H, E. If the trapezium be supposed to be made up of an infinite number of lines parallel to $A\,B$, each of these lines will be bisected by $E\,H$, and consequently the centre of gravity of the trapezium must be in the line $E\,H$.

Join $D\,E$ and take $F\,M = \frac{1}{3}\,E\,D$. Similarly, join $H\,B$, and take $H\,N = \frac{1}{3}\,H\,B$. Then M is the centre of gravity of the triangle $A\,B\,D$, and N that of the triangle $B\,C\,D$. Consequently the centre of gravity of the whole figure made up of these two triangles must be in the line $M\,N$; but it also lies in the line $E\,H$; therefore the intersection of these lines, or the point G is the centre of gravity required.



Let $G\,E$ be called x, and $G\,H$, y; and let $A\,B = a$, and $D\,C = b$. Draw $M\,m$, $N\,n$, parallel to $A\,B$ or $C\,D$. Then by similarity of triangles

$$M\,m : D\,H :: E\,M : E\,D$$

$$\therefore \quad M\,m = \tfrac{1}{3}\,D\,H \qquad \text{but } D\,H = \tfrac{1}{2}\,b,$$

$$\therefore \quad M\,m = \tfrac{1}{6}\,b.$$

Similarly

$$N\,n : E\,B :: H\,N : H\,B$$

$$\therefore \quad N\,n = \tfrac{1}{3}\,E\,B \qquad \text{but } E\,B = \tfrac{1}{2}\,a,$$

$$\therefore \quad N\,n = \tfrac{1}{6}\,a.$$

Also $\quad E\,m = \frac{1}{3}\,E\,H = \frac{1}{3}\,(x + y)$

and $\quad H\,n = \frac{1}{3}\,E\,H = \frac{1}{3}\,(x + y)$

Since the triangles $M\,G\,m$, $N\,G\,n$ are similar,

$$G\,m : G\,n :: Mm : Nn.$$

$$\therefore \quad G\,E - E\,m : G\,H - H\,n :: Mm : Nn.$$

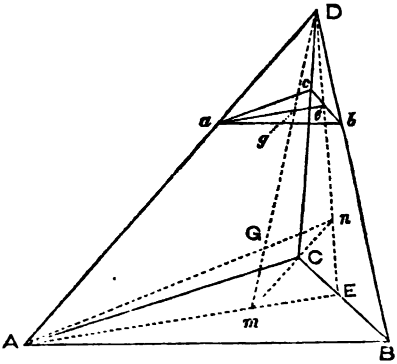
$$\therefore \quad x - \tfrac{1}{3}\,(x+y) : y - \tfrac{1}{3}\,(x+y) :: \tfrac{1}{6}\,b : \tfrac{1}{6}\,a$$

$$\therefore \quad 2\,x - y : 2\,y - x :: b : a$$

$$\therefore \quad x : y :: a + 2\,b : 2\,a + b.$$

*48. *To find the centre of gravity of a triangular pyramid.*

Let $A\,B\,C\,D$ be any triangular pyramid. Bisect $B\,C$ in E. Draw $A\,E$ and $D\,E$. Take $E\,m = \frac{1}{3}\,A\,E$, and $E\,n = \frac{1}{3}\,D\,E$. Join $D\,m$ and $A\,n$. These lines will intersect in a point G, which is the centre of gravity of the pyramid.



The pyramid may be conceived to be made up of an infinite number of planes parallel to $A\,B\,C$. Let $a\,b\,c$ be such a plane. Then since the parallel planes $a\,b\,c$, $A\,B\,C$, are intersected by a third plane $B\,C\,D$, the intersections $c\,b$, $C\,B$ are parallel. And since $c\,b$ is parallel to $C\,B$, $c\,b$ is bisected in e by the line $D\,E$, for by similarity of triangles

$$c\,e : C\,E :: D\,c : D\,C$$

and $$c\,b : C\,B :: D\,c : D\,C$$

$\therefore$ $$c\,e : c\,b :: C\,E : C\,B$$

$$\text{but } C\,E = \tfrac{1}{2}\,C\,B$$

$$\therefore\ c\,e = \tfrac{1}{2}\,c\,b.$$

The lines $a\,e$, $A\,E$, are parallel: for they lie in the same plane $A\,E\,D$, and in the parallel planes $a\,b\,c$, $A\,B\,C$. Then by similarity of triangles

$$\begin{array}{llcllcl} & e\,g & : & E\,m & :: & D\,e & : & D\,E \\ \text{and} & e\,a & : & E\,A & :: & D\,e & : & D\,E \\ \therefore & e\,g & : & e\,a & :: & E\,m & : & E\,A \\ \text{but} & & & E\,m & = & \tfrac{1}{3}E\,A & & \\ \therefore & & & e\,g & = & \tfrac{1}{3}\,e\,a, & & \end{array}$$

therefore g is the centre of gravity of the triangle $a\,b\,c$. In like manner it may be shewn that the centre of gravity of every plane parallel to $A\,B\,C$ is in the line $D\,m$. The centre of gravity of the whole pyramid must therefore be in the line $D\,m$. For similar reasons it must also be in the line $A\,n$. Therefore G the intersection of these two lines must be the centre of gravity of the pyramid.

Join $m\,n$ then, since $A\,m : m\,E :: D\,n : n\,E$, $m\,n$ is parallel to $A\,D$, and therefore

$$\begin{array}{lclcll} m\,n & : & A\,D & :: & m\,E & : A\,E \\ & \text{but} & m\,E & = & \tfrac{1}{3}\,A\,E & \\ & \therefore & m\,n & = & \tfrac{1}{3}\,A\,D. & \end{array}$$

The triangles $A\,G\,D$, $m\,G\,n$ are similar, therefore

$$\begin{array}{lclcll} m\,G & : & G\,D & :: & m\,n & : A\,D \\ & \therefore & m\,G & = & \tfrac{1}{3}\,G\,D & \\ & \therefore & m\,G & = & \tfrac{1}{4}.D\,m. & \end{array}$$

Therefore the centre of gravity of the pyramid is one fourth of the way up the line joining the centre of gravity of the base with the vertex.

*49. To find the centre of gravity of a pyramid whose base is any polygon.

Divide the polygon into triangles, and if planes be supposed to pass through the sides of these triangles and the vertex, the pyramid will be divided into triangular pyramids.

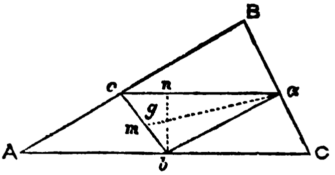
Let a plane be drawn parallel to the base at a distance equal to one fourth of the altitude of the pyramid, then Art. 48, the centre of gravity of each of the triangular pyramids will be in this plane. Therefore the centre of gravity of the whole pyramid will be in this plane.

Again, join the vertex with the centre of gravity of the base. The centre of gravity of every section parallel to the base will be in this line. Hence the centre of gravity of the whole pyramid will be in this line.

Therefore the centre of gravity is in the line joining the vertex with the centre of gravity of the base, at a point whose distance from the base is one fourth of this line.

*50. *To find the centre of gravity of the perimeter of a triangle.*

Let $A\ B\ C$ be the given triangle. Join $a\ b\ c$ the points of bisection of the three sides. Draw $a\ m$, $b\ n$ bisecting the



angles $c\ a\ b$, $a\ b\ c$ respectively. The point g the intersection these two lines will be centre of gravity of the perimeter of the triangle $A\,B\,C$. For the three lines $A\,B$, $B\,C$, $C\,A$ may be conceived to act at the points c, a, b, with forces represented by their several lengths. The triangles $a\,b\,c$, $A\,B\,C$ are similar, and therefore

$$AB : BC : CA :: ab : bc : ca.$$

Hence

$$\text{The force at } c : \text{the force at } b :: a\,b : a\,c.$$

But since $a\,m$ bisects the angle $b\,a\,c$

$$\begin{array}{lrcl} & b\,m : m\,c & :: & a\,b : a\,c \\ \therefore & \text{force at } c : \text{force at } b & :: & b\,m : m\,c. \end{array}$$

Consequently the resultant of the forces at b and c passes through m, and may be represented in value by $a\,b + a\,c$. The resultant of this force, and the force at a represented by $b\,c$, must pass through g, for

$$\begin{array}{lrcl} & g\,m : g\,a & :: & m\,b : a\,b \\ \text{but} & m\,b : a\,b & :: & b\,c : a\,b + a\,c \\ \therefore & g\,m : g\,a & :: & b\,c : a\,b + a\,c. \end{array}$$

The point g is also the centre of the circle inscribed in $a\,b\,c$. Hence the centre of gravity of the perimeter of a triangle is the centre of the circle inscribed in the triangle formed by joining the bisections of its sides.

51. If the magnitudes and centres of gravity of the several parts of a body of uniform density are known, the centre of gravity of the whole body can be found by the proposition given in Art. 32.

For let M be the total magnitude of the body, and M_1 M_2 M_3, &c., the magnitudes of its several parts. But instead of these bodies, we may suppose forces proportional to their weights to act at their respective centres of gravity. Since the density of the body is uniform, these weights will be proportional to the magnitudes. Consequently if h be the distance of the centre of gravity of the whole body from any plane, and h_1, h_2, h_3, &c., the known distances of the centres of gravity of the several parts from the same plane;

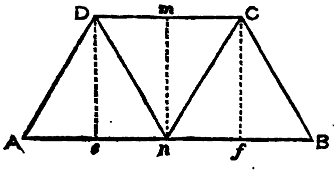
$$M h = M_1 h_1 + M_2 h_2 + M_3 h_3 + \dots$$

$\therefore$

$$h = \frac{M_1 h_1 + M_2 h_2 + M_3 h_3 + \dots}{M}$$

*52. To find the centre of gravity of half a regular hexagon.

Let $A B C D$ be half a regular hexagon. Bisect $A B$ in n, and join $D n$, $C n$. Then the given figure is divided into three equilateral triangles, whose centres of gravity are known.



Bisect $D C$ in m and join $m n$. Then the centre of gravity must evidently be in this line, and will consequently be found if we can determine its distance from $A B$.

Let the side of the hexagon $= a$, and let $D e = c$.

Then the magnitude of each of the triangles is $\frac{a c}{2}$, and of the whole figure $\frac{3 a c}{2}$. The distance of the centre of gravity of the triangles $A\,D\,n$, $B\,C\,n$ from $A\,B$, or which is the same thing, its distance from a plane drawn through $A\,B$ perpendicular to the plane of the paper, is $\frac{c}{3}$. The distance of centre of gravity of the triangle $C\,D\,n$ from $A\,B$ is $\frac{2\,c}{3}$. Hence if h denote the distance of the centre of gravity of the whole figure from $A\,B$, then by the preceding Art.

$$h = \frac{\frac{a\,c}{2}\cdot\frac{c}{3} + \frac{a\,c}{2}\cdot\frac{c}{3} + \frac{a\,c}{2}\cdot\frac{2\,c}{3}}{\frac{3\,a\,c}{2}}$$

$$= \frac{4\,c}{9}$$

*53. *Let the upper face of a cube be the base of a square pyramid, to find the centre of gravity of the whole figure, the density being uniform.*

Let $a =$ the edge of the cube, and $b =$ the height of the pyramid. The centre of gravity must evidently lie in the line joining the vertex of the pyramid with the centre of the base of the cube. Let the base of the cube be the plane from which the distances of the centres of gravity are reckoned.

The distance of the centre of gravity of the cube from this plane $= \frac{a}{2}$, and the distance of the centre of gravity of the pyramid from the same plane $= a + \frac{b}{4}$

The content of the cube $= a^3$

.... pyramid $= \frac{a^2 b}{3}$

Therefore, if h denote the distance of the centre of gravity of the whole figure from the same plane, by Art. 51.

$$h = \frac{a^3 \cdot \frac{a}{2} + \frac{a^2 b}{3}\left(a + \frac{b}{4}\right)}{a^3 + \frac{a^2 b}{3}}$$

$$= \frac{6\, a^2 + 4\, a\, b + b^2}{12\, a + 4\, b}$$

$$= \tfrac{1}{4} \cdot \frac{6\, a^2 + 4\, a\, b + b^2}{3\, a + b.}$$

If $b = a$, that is, if the height of the pyramid be the same as that of the cube

$$h = \frac{11}{16}\, a$$

EXAMPLES.

1. Find the centre of gravity of half a regular octagon.

2. In a quadrilateral figure $A\,B\,C\,D$, let $A\,B = A\,D$ and $C\,B = C\,D$, determine geometrically the position of the centre of gravity.

3. If from a square $A\ B\ C\ D$, whose diagonals intersect in O, the portion $A\ O\ D$ be taken away; the distance of the centre of gravity of the remainder from O is (a being the side of the square) $= \frac{1}{9}\, a$.

4. Upon a side of a square describe an equilateral triangle, and let VN be the line joining the vertex with the bisection of the base, then if G be the centre of gravity of the whole figure,

$$GN : VN :: 4-\sqrt{3} : 4+\sqrt{3}.$$

5. If from any pyramid whose height is a, a pyramid whose height is b be cut off by a plane parallel to the base, shew that the distance from the base of the centre of gravity of the frustrum is equal to

$$\frac{a-b}{4} \cdot \frac{a^2+2ab+3b^2}{a^2+ab+b^2}$$

6, Let two spheres of the same density, whose radii are a and b respectively, touch one another externally, the distance of their centre of gravity from the centre of the sphere whose radius is a is

$$\frac{b^3}{a^2-ab+b^2}$$

and its distance from the centre of the sphere whose radius is b is

$$\frac{a^3}{a^2-ab+b^2}.$$

CHAPTER V.

ON THE MECHANICAL POWERS.

54. A *Machine* is an instrument, by the agency of which one force can resist or overcome another force not immediately opposite to it in direction.

55. The simplest of such instruments are denominated the *Mechanical Powers*. These are, the Lever, the Wheel and Axle, Toothed Wheels, the Pulley, the Inclined Plane, the Wedge, and the Screw. These differ from each other more in their structure than in the principles of their operation, for when in equilibrium, the Wheel and Axle, the Toothed Wheel, and the Pulley may be reduced to the Lever; and the Wedge and the Screw are but modifications of the inclined plane.

The two forces which act upon either of these simple machines, are for the sake of distinction, called the *Power* and the *Weight*, the latter always denoting the force to be resisted or overcome.

THE LEVER.

56. The Lever is an inflexible bar, capable of free motion about a fixed axis, called the fulcrum. Unless the contrary be stated, the lever is usually supposed to be without weight.

Levers are of three kinds, according to the relative position of the power, weight and fulcrum.

When the fulcrum is between the power and the weight, the lever is of the first kind.

When the weight is between the fulcrum and the power, the lever is of the second kind.

When the power is between the fulcrum and the weight, the lever is of the third kind.

The beam of a balance is a lever of the first kind, an oar is a lever of the second kind, and the treadle of a lathe is a lever of the third kind.

Scissors are double levers of the first kind, nut-crackers are double levers of the second kind, and spring shears are double levers of the third kind.

57. *To find the condition of equilibrium in the lever, when the power and the weight act in parallel directions.*

It is evident that there will be equilibrium, if the resultant of the two forces passes through the fulcrum. Let P represent the power and W the weight; let a be the distance from the fulcrum at which P acts, and b the distance at which W acts. Then since P and W are by hypothesis parallel forces, their resultant will pass through the fulcrum if (Art. 24.)

$$P a = W b,$$

whether the fulcrum be between the forces as in levers of the first kind, or without them as in levers of the second and third kinds.

Hence, when the power and the weight are parallel forces, the condition of equilibrium in a lever of either kind, is that

The Power × its distance from fulcrum = the Weight × its distance from the fulcrum.

58. *Ex.* A weight of 20lbs. is suspended at a distance of 3 inches from the fulcrum of a lever of the first kind, where must a power of 4lbs. be suspended, in order that there may be equilibrium?

Let x = the distance sought, then

$$4x = 20 \times 3 = 60$$

$$\therefore \quad x = 15 \text{ inches.}$$

59. ***To find the condition of equilibrium in a lever acted upon by parallel forces, when the weight of the lever itself is regarded.***

Let P denote the power and W the weight. Let a = the length of the arm at which P acts, and b = the length of the arm at which W acts. Let w = the weight of the lever, and c = the distance of its centre of gravity from the fulcrum.

If the lever be of the first kind, and if the centre of gravity be on the same side of the fulcrum as P, then, Art. 29, the resultant will pass through the fulcrum, if

$$P a + w c = W b.$$

The same condition will also apply to levers of the second and third, if the weight of the lever act in the same direction as the power. If, however, the weight of the lever act in the same direction as the weight, the condition of equilibrium will be

$$P a = W b + w c.$$

60. *Ex.* The two arms of a lever of uniform thickness and density, are 7 in. and 3 in. respectively in length; a weight of 12lbs. is suspended from the shorter arm, what power will support it, the weight of the lever being 4lbs?

Since the lever is of uniform thickness and density, its centre of gravity is at half its length, and therefore at a distance of 2 in. from the fulcrum. Let P be the force required, then

$$P \times 7 + 4 \times 2 = 12 \times 3$$

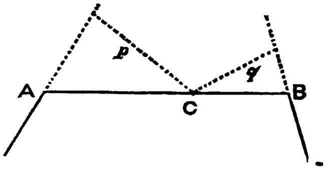
or

$$7P + 8 = 36$$

$\therefore$

$$P = 4\text{lbs.}$$

61. *To determine the condition of equilibrium in a lever when the power and weight act in any directions whatever.*



Let P and Q be forces acting in any direction at the extremities A and B of any lever. From the fulcrum C let fall perpendiculars upon the directions of the forces, and let their lengths be p and q respectively. Then, Art. 21, the resultant of P and Q passes through C, if

$$Pp = Qq,$$

which is consequently the condition of equilibrium required.

62. In the preceding article it is not necessary that the lever be a straight one. Whatever the relative positions of A, B and C, the same condition is sufficient. For if p and q be the lengths of the perpendiculars let fall from C upon the directions of the forces, and if

$$Pp = Qq,$$

the resultant will pass through C, and consequently whatever be the shape of the lever there will be equilibrium.

63. *To determine the pressure upon the fulcrum of a lever.*

The pressure upon the fulcrum is in every case the resultant of the forces acting upon the lever. When these are parallel and act in the same direction the resultant equals their sum,—if in opposite directions the resultant equals their difference. Hence the pressure upon the fulcrum,

in levers of the first kind, is $P + W$,
.... second...... $W—P$,
.... third ...... $P—W$.

If the weight of the lever itself be regarded, it must be added to the power or to the weight according as it acts in conjunction with the one or with the other. Then if $w =$ the weight of the lever, the pressure upon the fulcrum,

in levers of the first kind, is $P + W + w$,
.... second...... $W—\overline{P + w}$ or $\overline{W + w}—P$,
.... third ...... $\overline{P + w}—W$ or $P—\overline{W + w}$.

64. An ordinary balance is, mechanically considered, simply a lever of the first kind provided with pans suspended from its extremities. When the arms of the balance are equal, a body placed in one of the pans will be balanced by an equal weight in the other. When however the arms are unequal, the weight in one pan is not equal to that of the body in the other.

65. *To determine the true weight of a body by a false balance.*

Let the true weight of the body be w. When placed in one pan of the false balance, let the body be balanced by a weight a, and when placed in the other pan by a weight b. Let x and y be the unknown arms of the balance.

In the first case, w acting at the arm x was balanced by the weight a acting at the arm y; therefore, Art. 57,

$$w\,x = a\,y.$$

In the second case, w acting at the arm y was balanced by the weight b acting at the arm x; and therefore

$$w\,y = b\,x.$$

Consequently $\quad w^2\,x\,y = a\,b\,x\,y.$

$\therefore \quad w^2 = a\,b \qquad \text{or } w = \sqrt{a\,b}.$

That is, the true weight is a mean proportional between the two false weights.

66. *To graduate the common Steel Yard.*

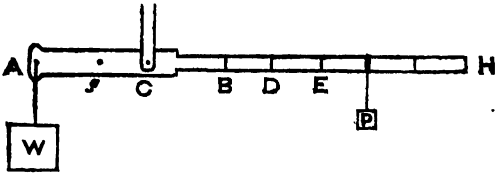
The common Steel Yard is a lever of the first kind with very unequal arms, provided with a moveable weight P, that may be suspended from any point in the longer arm. The longer arm is graduated, and the weight of any body suspended from the shorter arm is indicated by the point at which P must be placed in order to balance it.

Let a = length of the shorter arm, and w = weight of the beam, and let the centre of gravity of the beam be at the point g in the shorter arm.

Let B be the point at which P must be placed in order to balance the weight of the beam, then if $C\,B = m$ and $g\,C = x$

$$w\,x = P\,m.$$

Let a weight of 1℔. suspended from A be balanced when P is suspended from D, then if successive distances, each equal to $B\ D$ be marked along the arm $B\ H$, they will denote



the positions of P corresponding to successive additions of 1℔. to the weight. For let $B\ D = b$, then

$$1 \times a + w\,x = P\,(b + m)$$
$$\therefore \qquad a = P\,b \qquad \text{since } w\,x = P\,m.$$

Now let P be suspended from the next division E, that is at a distance from the fulcrum $= m + 2\,b$, then if W be the corresponding weight

$$W\,a + w\,x = P\,(m + 2\,b)$$
$$\therefore \qquad W\,a = 2\,P\,b$$
$$= 2\,a \qquad \text{since } a = P\,b.$$
$$\therefore \qquad W = 2\text{lbs.}$$

And generally if P be suspended at a distance $= m + n\,b$, and be balanced by a weight W, then W must $= n$ lbs.,

$$\text{for} \qquad W\,a + w\,x = P\,(m + n\,b)$$
$$= P\,m + n\,P\,b$$
$$\therefore \qquad W\,a = n\,P\,b = n\,a$$
$$\therefore \qquad W = n.$$

If the centre of gravity of the beam does not fall between A and C, but falls upon the other side of the fulcrum, at a distance x from C, then the steel yard will serve to weigh those bodies only whose weight is greater than $\frac{w\,x}{a}$. Equal increments of weight are, however, still denoted by divisions at equal distances.

Let p be some whole number of lbs. greater than $\frac{w\,x}{a}$, and let p lbs. suspended from A be balanced by the moveable weight P at B, and let $p + 1$ lbs. be balanced by P at D, then if $D\,E = B\,D$, P at E will balance $p + 2$ lbs. and so on. For as before, let $B\,C = m$, $B\,D = b$, and $A\,C = a$, then by the hyp.

$$p\,a = P\,m$$

and $$(p+1)\,a = P\,(m+b)$$

$\therefore$ $$a = P\,b.$$

Let W be the weight balanced by P at the n th division from B,

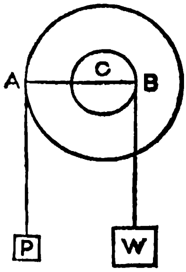
then
$$\begin{aligned} W\,a &= P\,(m + n\,b) \\ &= P\,m + n\,P\,b \\ &= p\,a + n\,a \end{aligned}$$

$\therefore$ $$W = p + n.$$

THE WHEEL AND AXLE.

67. The wheel and axle consists of a cylinder or axle firmly fixed to a wheel, and having a common axis with it. The weight is attached to a cord passing round the axle, and the power to a cord passing round the wheel.

Let the figure represent a section of such a machine. Let C be a point in the common axis, $C A$ the radius of the wheel and $C B$ the radius of the axle.



Let P be the power and W the weight. These may be regarded as two parallel forces, and if in equilibrium their resultant must pass through C, and therefore we must have

$$P \times A C = W \times B C,$$

that is

$$P \times \text{radius of wheel} = W \times \text{radius of axle},$$

or

$$\frac{P}{W} = \frac{\text{radius of wheel}}{\text{radius of axle}}$$

which is consequently the ratio of the power and weight when there is equilibrium.

TOOTHED WHEELS.

68. A toothed wheel is a circular plate of wood or metal, having its circumference indented or cut into equal teeth all the way round. If two such wheels, having their teeth of the same magnitude and at the same distance apart, be so placed

that a tooth of one may lie between any two of the other, then if one of them be turned round by any means the other will be turned round also.

If the teeth be large, the amount and direction of the pressure which the one wheel exerts upon the other, will vary considerably as the wheels revolve; but if the teeth be small in comparison with the size of the wheel, this variation may be disregarded, and the mutual pressure may without any great error be treated as constant in magnitude and direction.

69. *To find the ratio of the power and weight in toothed wheels when the teeth are small.*

Let the power and the weight both act at axles of equal radius, and let c be this radius. Let a be the radius of P's wheel, and b the radius of W's wheel.

Since the teeth are small, the pressure of the one wheel upon the other may be regarded as constant, and as acting in the direction of the common tangent to the two wheels. Let Q denote this pressure. Then since P and Q are in equilibrium, by Art. 67.

$$P c = Q a,$$

and since W and Q are in equilibrium

$$W c = Q b$$

$\therefore$ multiplying crosswise $P Q b c = W Q a c$

$$\therefore \quad P b = W a$$

$$\text{or} \quad P : W :: a : b.$$

But since the teeth in each wheel are of the same magnitude and at the same distance apart, the number of teeth in each wheel will be proportional to the circumference, and consequently to the radius. Therefore,

P : W :: number of teeth in P's wheel : number of teeth in W's wheel.

THE PULLEY.

70. A pulley consists of a small wheel, which moves freely about an axis, and allows a cord to pass over any part of its circumference.

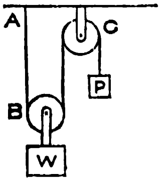
Unless it be otherwise stated, the wheel is supposed to revolve without friction, and the cord to be perfectly flexible.

No mechanical advantage is gained by a fixed pulley; for as the tension in every part of the cord is the same, if a weight W be suspended at one extremity, an equal weight must be applied at the other to maintain equilibrium. Hence in this case

$$P = W$$

The effect of a fixed pulley is simply to change the direction of a force.

71. *To find the ratio of the power and weight in a single moveable pulley when the cords are parallel.*



Let the weight W be attached to the moveable pulley B, and let B be sustained by a cord $A\ B\ C$, one extremity of which is fastened at A, and the other after passing over the fixed pully C sustains the power P.

Let there be equilibrium:—then the weight W is sustained by the tension in $B\,A$ and the tension in $B\,C$; but since the cords are parallel, these tensions may be regarded as two parallel forces, and therefore W must equal their sum. But the tension of the cord is the same throughout, and is equal to P,

$$\therefore \qquad W = 2\,P \quad \text{or} \quad P = \tfrac{1}{2}\,W.$$

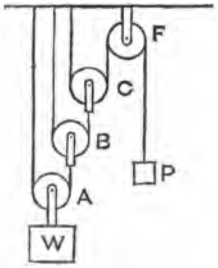
72. The following combinations of pullies are termed respectively, the first, second and third system of pullies.

In the first system of pullies, each pulley hangs by a separate cord, one end of which is fastened to a fixed beam, and the other to the pulley above it. See fig. Art. 73.

In the second system of pullies, the same cord passes round all the pullies, which are arranged in two blocks, one of which is fixed and the other bears the weight.

In the third system of pullies, each cord is attached to the weight: see fig. in Art. 76.

73. *To find the ratio of the power and weight in the first system of pullies, when the weight of the pullies is disregarded.*



Let A, B, C be three moveable pullies, and let P and W be in equilibrium. By Art. 71,

$$\text{tension in } A\,B = \tfrac{1}{2}\,W$$

and $$\text{tension in } B\,C = \tfrac{1}{2} \text{ tension in } A\,B = \frac{1}{2^2}\,W$$

Similarly $$\text{tension in } C\,F = \tfrac{1}{2} \text{ tension in } B\,C = \frac{1}{2^3}\,W$$

But by Art. 70, tension in $C\;F = P$, therefore when there are three moveable pullies

$$P = \frac{1}{2^3}\,W \quad \text{or} \quad W = 2^3\,P.$$

In like manner if there are n moveable pullies

$$P = \frac{1}{2^n}\,W \quad \text{or} \quad W = 2^n\,P.$$

*74. *To find the ratio of the power and weight in the first system of pullies, when the weight of the pullies is regarded.*

The total weight acting at each pulley is the tension of the string attached to the block, together with the weight of the block. Let each of the moveable pullies be of the weight w, then

$$\text{weight at } A = W + w$$

$$\text{tension in } A\,B = \tfrac{1}{2} \text{ weight at } A = \frac{W}{2} + \frac{w}{2}$$

$$\therefore \quad \text{weight at } B = \frac{W}{2} + \frac{w}{2} + w$$

$$\text{tension in } B\,C = \tfrac{1}{2} \text{ weight at } B = \frac{W}{2^2} + \frac{w}{2^2} + \frac{w}{2}$$

$$\therefore \quad \text{weight at } C = \frac{W}{2^2} + \frac{w}{2^2} + \frac{w}{2} + w$$

$$\text{tension in } C\,F = \tfrac{1}{2} \text{ weight at } C = \frac{W}{2^3} + \frac{w}{2^3} + \frac{w}{2^2} + \frac{w}{2}$$

but $P =$ tension in $C\,F$

$$\therefore \quad P = \frac{W}{2^3} + \frac{w}{2^3} + \frac{w}{2^2} + \frac{w}{2}$$

$$= \frac{W}{2^3} + \frac{w}{2^3}\left(1 + 2 + 2^2\right)$$

$$= \frac{W}{2^3} + w\,\frac{2^3-1}{2^3}$$

In like manner if there are n moveable pullies

$$P = \frac{W}{2^n} + w\,\frac{2_n-1}{2^n} \quad \text{or} \quad W = 2^n\,P - w\,(2_n-1).$$

75. *To find the ratio of the power and weight in the second system of pullies.*

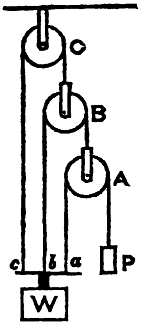
Since there is but one cord, and P is attached to one extremity of it, the tension in every part is equal to P. Hence if n be the number of the portions of this cord in contact with the lower block, the weight supported will be $n\,P$; therefore in this system

$$W = n\,P \quad \text{or} \quad P = \frac{1}{n}\,W.$$

If the weight of the lower block be w, the total weight supported is $W + w$; therefore if the weight of the block be regarded

$$W + w = n\,P \quad \text{or} \quad P = \frac{1}{n}\,(W + w).$$

76. *To find the ratio of the power and weight in the third system of pullies.*



Let there be three pullies A, B, C; then

$$\begin{aligned}\text{tension in } A\,a &= P\\ \text{tension in } B\,b &= \text{pressure on } A = 2\,P\\ \text{tension in } C\,c &= \text{pressure on } B = 2^2\,P.\end{aligned}$$

But as the weight is supported by the tensions in the three cords $A\,a$, $B\,b$ and $C\,c$, and since the cords are parallel, W must equal the sum of the tensions; therefore

$$\begin{aligned}W &= P + 2\,P + 2^2\,P\\ &= P\,(1 + 2 + 2^2)\\ &= P\,(2^3-1).\end{aligned}$$

Similarly if there are n pullies, $W = P\,(2^n-1)$.

*77. *To find the ratio of the power and weight in the third system of pullies, when the weight of the pullies is regarded.*

Let each of the pullies be of the weight w, then

$$\begin{aligned}\text{tension in } A\,a &= P\\ \text{tension in } B\,b &= 2\,P + w\\ \text{tension in } C\,c &= 2^2\,P + 2\,w + w.\end{aligned}$$

$$\therefore \qquad W = P\,(2^3-1) + w\,(2^2-1) + w.$$

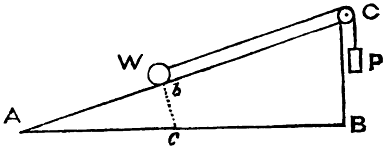
Similarly if there be n pullies

$$W = P(2^n - 1) + w(2^{n-1} - 1) + w(2^{n-2} - 1) + \ldots + w(2^2 - 1) + w$$
$$= P\,(2^n - 1) + w\,(2_n - 1 - n).$$

THE INCLINED PLANE.

78. *To find the ratio of the power and weight in the inclined plane when the power acts parallel to the plane.*

Let W be a weight resting upon an inclined plane at the point b, and sustained there by a weight P, with which it is connected by means of a cord $b\ C$. Through b draw $b\ c$ perpendicular to $A\ C$; then the three sides of the triangle $A\ b\ c$ are severally perpendicular to the three forces acting upon W, and keeping it at rest, for



$b\ c$ is perpendicular to the direction of the power P,
$A\ c$ .... the direction of gravity.
$A\ b$ .... the re-action of the plane.

∴ by Art. 15.

$$P : W :: b\,c : A\,c.$$

But the triangles $A\ b\ c$, $A\ B\ C$ are similar, and therefore

$$b\,c : A\,c :: B\,C : A\,C$$
$$\therefore \quad P : W :: B\,C : A\,C,$$

or $$\frac{P}{W} = \frac{\text{height of the plane}}{\text{length of the plane}}$$

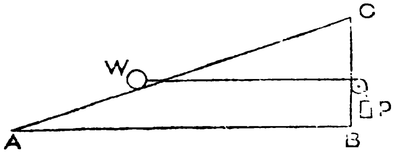
In a similar manner the pressure which W exerts upon the plane $A\,C$ may be determined, for

$$\begin{aligned} \text{pressure on the plane} : W &:: A\,b : A\,c \\ &:: A\,B : A\,C. \end{aligned}$$

$$\therefore \quad \frac{\text{pressure on the plane}}{W} = \frac{\text{base of the plane}}{\text{length of the plane.}}$$

79. *To find the ratio of the power and weight in the inclined plane, when the power acts horizontally.*

In this case W is kept at rest by three forces, viz, the force of gravity acting vertically, the power acting horizontally, and the re-action of the plane acting perpendicularly to the plane.



The sides of the triangles $A\ B\ C$ are severally perpendicular to the directions of these forces, and therefore, Art. 15;

$$P : W :: B\,C : A\,B,$$

or $$\frac{P}{W} = \frac{\text{height of the plane}}{\text{base of the plane.}}$$

For similar reasons,

$$\text{pressure on the plane} : W :: A\,C : A\,B$$

or $$\frac{\text{pressure on the plane}}{W} = \frac{\text{length of the plane}}{\text{base of the plane.}}$$

80. *Two weights connected by a cord which passes over a pulley at the summit, rest upon a double inclined plane; to find their ratio when in equilibrium.*

Let l and l' be the lengths of the two planes supporting respectively the weights W and P. Let h be the common height of the planes. If t denote the tension in the cord, then since W is supported on an inclined plane by a force t acting along the plane, by Art. 78,

$$\frac{t}{W} = \frac{h}{l}$$

Similarly

$$\frac{t}{P} = \frac{h}{l'}$$

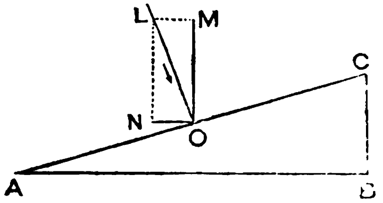
$$\therefore \quad \frac{P}{W} = \frac{l'}{l},$$

that is, the two weights are in the same ratio as the lengths of the planes on which they respectively rest.

81. *To determine the vertical and horizontal pressures, when a force P acts perpendicularly to a given inclined plane.*

Let $A\,C$ be the given inclined plane, whose height $B\,C = h$, whose length $A\,C = l$, and whose base $A\,B = b$.

Let a force P act at the point O, in the direction $L\,O$ perpendicular to $A\,C$.



Through O draw the vertical line $O\,M$, and the horizontal line $O\,N$. Complete the rectangle $M\,N$. Then by the parallelogram of forces, if $O\,L$ represent the force P, P may be resolved into two forces represented in magnitude and direction by $O\,M$ and $O\,N$ respectively Therefore

$$\text{vertical pressure} : P :: O\,M : O\,L.$$

But the triangles $O\,M\,L$, $A\,C\,B$ are similar, therefore

$$\text{vertical pressure} : P :: A\,B : A\,C$$

$$\therefore \quad \text{vertical pressure} = P\,\frac{AB}{AC} = P\,\frac{b}{l}.$$

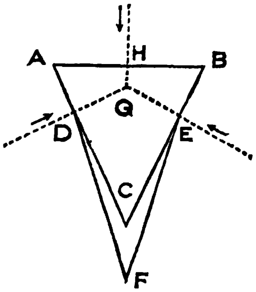
Similarly,—horizontal pressure : $P :: ON : OL :: BC : AC$

$$\therefore \quad \text{horizontal pressure} = P\,\frac{BC}{AC} = P\,\frac{h}{l}.$$

Hence, if $A\,B\,C$ were free to slide along a smooth horizontal plane, $P\,\frac{h}{l}$ is the horizontal force that must be applied to keep it at rest, and $P\,\frac{b}{l}$ will denote the pressure upon the horizontal plane.

THE WEDGE.

82. *To find the ratio of the power and the resistance in an isosceles wedge.*



Let $A\ B\ C$ be the section of an isosceles wedge introduced into the cleft $D\ F\ E$, and let the points D, E be similarly situated on the two sides of the wedge. The resistance on each side of the wedge will be the same, and if $R =$ the total resistance, the resistances at D and E will each $= \frac{1}{2} R$; and they act perpendicularly to the sides of the wedge. Let a power P act at the point H, the centre of the back of the wedge. The directions of these three forces when produced will meet in a point G, they may therefore be considered as three forces acting upon a point, and in equilibrium. The sides of the triangle $A\ B\ C$ are severally perpendicular to the directions of the three forces, and therefore, Art. 15.

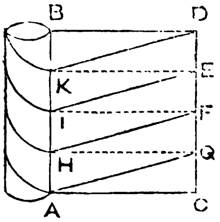
$$P : \tfrac{1}{2} R :: A B : A C$$

$$\therefore \quad P : R :: \tfrac{1}{2} A B : A C,$$

that is, the power is to the total resistance, as half the back of the wedge is to the side of the wedge.

THE SCREW.

83. Let *A B* be a cylinder, and *A D* a rectangle, whose base *A C* is equal to the circumference of the cylinder. Let the side *C D* be divided into any number of equal parts. Join *A G*, and through the points *F*, *E*, *D*, draw *F H*, *E I*, *D K* parallel respectively to *A G*. Then if the rectangle *A D* be wrapped round the cylinder, the parallel lines *A G*, *H F*, &c., will trace out the continued spiral line called the screw.



Hence any resistance to be overcome by a screw, may be regarded as a weight resting upon an inclined plane, whose base is equal to the circumference of the cylinder, and whose height is equal to the distance between the threads of the screw.

When the screw is used for mechanical purposes, the power is always applied in a direction perpendicular to the axis of the cylinder, and consequently parallel to the base of the inclined plane forming the screw line, and therefore [*]

$$\frac{P}{W} = \frac{\text{height of the inclined plane}}{\text{base of the plane}}$$
$$= \frac{\text{distance between the threads}}{\text{circumference of the cylinder.}}$$

CHAPTER VI.

STATICAL PROBLEMS.

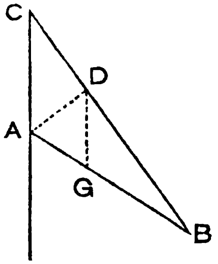
*84. The resistance which a perfectly smooth surface presents to any body in contact with it, must act in a direction perpendicular to the surface, for there can be no reason why it should act on one side of the perpendicular more than upon another. Hence when one point of any body presses against a perfectly smooth surface, or when the smooth surface of any body rests upon a fixed point, the resistance in every case, acts in the direction of the perpendicular to the surface drawn through the point of contact.

*85. If three forces (not being parallel forces) acting at any points in a rigid body are in equilibrium, there must be a point in which their directions, if produced will meet. For let the directions of any two be produced till they meet, then their resultant must pass through this point, and since there is equilibrium, the third force must also pass through this point, (Art. 8.) Consequently the relations which have been established between three forces acting upon a point when in equilibrium, exist also between three forces in equilibrium acting at any points in a rigid body.

*86. When more than three forces act at various points of a rigid body, many problems of equilibrium may be conveniently solved, if all the forces can be brought to act at any two points in the body. For then, since there is equilibrium, the resultant of one set of forces must be equal and opposite to the resultant of the other set, and consequently the resultant of each set must act along the line joining the two points. And if at each point a force equal and opposite to the resultant be introduced, equilibrium will subsist between each set of forces separately.

*87. *One extremity of a beam $A B$ is fastened by a pivot at A, and a cord tied to the other extremity B, is fastened to a point C lying above A in the same vertical line, required the tension in the string.*

Let W = weight of the beam, acting at its centre of gravity G. Let $A B = l$, $A G = a$, $A C = h$, and $B C = c$.



The three forces acting upon the beam are the weight of the beam, the tension of the string, and the resistance of the pivot. Through G draw the vertical line $G D$. Join $A D$, and let T be the tension of the string. T acts along $C B$, and W along $G D$, consequently the resultant of T and W must pass through D. But since there is equilibrium, this resultant must be equal and opposite to the third force, viz., the resistance of the pivot, and consequently $A\,D$ is the direction in which this resistance acts.

The sides of the triangle $A\,C\,D$ are severally parallel to the directions of the three forces, and therefore

$$T : W :: C\,D : A\,C$$

$$:: C\,D : h \qquad \therefore\ T = W\frac{C\,D}{h}$$

But $C\,D : C\,B :: A\,G : A\,B$

that is $C\,D : c :: a : l \qquad \therefore\ C\,D = \dfrac{a\,c}{l}$

$$\therefore \qquad T = W\frac{a\,c}{h\,l}$$

*88. In the preceding problem the resistance of the peg can be easily determined when the beam is of uniform thickness and density. For the centre of gravity of the beam will then lie midway between its extremities, and consequently $A\,G = \frac{1}{2}\,A\,B$, that is $a = \frac{1}{2}\,l$; therefore also $C\,D = \frac{1}{2}\,c$.

Hence

$$2\,(A\,D)^2 + 2\,(C\,D)^2 = (A\,C)^2 + (A\,B)^2$$

$$\therefore \quad 2\,(A\,D)^2 + \tfrac{1}{2}\,c^2 = h^2 + l^2$$

$$\therefore \quad A\,D = \tfrac{1}{2}\sqrt{2\,(h^2 + l^2) - c^2}.$$

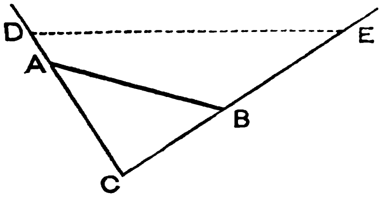
Let R be the resistance of the peg, then

$$R : W :: A\,D : A\,C$$

$$:: \tfrac{1}{2}\sqrt{2\,(h^2 + l^2) - c^2} : h$$

$$\therefore \quad R = \frac{W}{2h}\sqrt{2\,(h^2 + l^2) - c^2}.$$

*89. *The extremities of a beam rest upon two perfectly smooth inclined planes, required the pressure on the planes when there is equilibrium.*



Let $A\,B$ be a beam resting at A and B upon the planes $C\,D$ and $C\,E$. Let $D\,E$ be a horizontal line. Let $C\,D = a$, $C\,E = b$, and $D\,E = c$. Let $W =$ the weight of the beam.

Let P be the pressure on $C\,D$ and Q the pressure on $C\,E$. These pressures are perpendicular respectively to the planes: hence the sides of the triangle $C\,D\,E$ are severally perpendicular to the directions of the three forces acting on the beam and in equilibrium; therefore (Arts. 15 and 85).

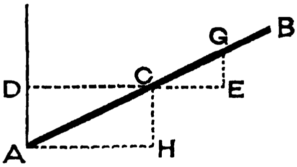
$$P : W :: a : c \quad \therefore \quad P = W\frac{a}{c}.$$

and
$$Q : W :: b : c \quad \therefore \quad Q = W\frac{b}{c}.$$

*90. *A beam rests freely upon a peg, with one end against a smooth vertical wall, required the position of equilibrium and the pressures on the wall and peg.*

Let $A\,B$ be a beam resting upon the peg C and against the wall $A\,D$. Let G be the centre of gravity of the beam, and

W its weight. Draw DE, AH perpendicular to AD, and GE, CH parallel to AD. Let $GA = a$, and $CD = b$. When there is equilibrium, let $CA = x$.



The resistance of the wall is equal and opposite to the pressure of the beam against it. Let R denote this resistance and Q the resistance of the peg. The beam then is acted on by three forces, by R acting perpendicularly to the wall, by Q acting perpendicularly to the beam, and by W acting vertically. The sides of the triangle ACD are severally perpendicular to the directions of these forces, and therefore since there is equilibrium, (Art. 15),

$$Q : W :: AC : CD$$

$$:: x : b \quad \therefore \; Q = W\frac{x}{b}. \qquad \text{(i.)}$$

and

$$R : W :: AD : CD$$

$$:: \sqrt{x^2-b^2} : b \quad \therefore \; R = W\frac{\sqrt{x^2-b^2}}{b}. \qquad \text{(ii.)}$$

In order that equilibrium may subsist, the resultant of R and W must pass through C; therefore Art. 21,

$$R \times CH = W \times CE.$$

But $CH = AD = \sqrt{x^2-b^2}$; and by similarity of triangles $CE : CG :: CD : AC$, that is $CE : a-x :: b : x$, or $CE = b\,\frac{a-x}{x}$; hence

$$R\sqrt{x^2-b^2} \quad = \quad W b\,\frac{a-x}{x}. \qquad \text{(iii.)}$$

The equations i, ii, and iii are sufficient to determine the three unknown quantities R, Q and x. For substituting in equation iii, the value of R given by equation ii, we have

$$x^2-b^2 = b^2\cdot\,\frac{a-x}{x}$$

$$\therefore \qquad x^3 = b^2 a \quad \text{or} \quad x = b\sqrt[3]{\frac{a}{b}}.$$

Substituting this value of x in equations i and ii, we have

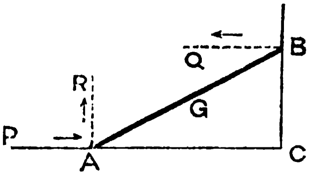
$$Q = W\sqrt[3]{\frac{a}{b}} \quad \text{and} \quad R = W\sqrt{\left(\frac{a}{b}\right)^{\frac{2}{3}}-1}.$$

*91. In the preceding example, if the beam instead of resting freely upon the peg C, were fixed by a pivot at any point C, the only condition of equilibrium is, that the resultant of R and W pass through C. Therefore, if $CA = c$, the pressure on the wall is obtained immediately from equation iii;—whence

$$R\sqrt{c^2-b^2} = W b\,\frac{a-c}{c} \quad \therefore \quad R = W\,\frac{b}{c}\cdot\frac{a-c}{\sqrt{c^2-b^2}}.$$

*92. *One end of a beam AB rests against a smooth vertical wall B C, and the other upon a smooth horizontal plane A C; required the horizontal force to be applied at the lower extremity, in order that the beam may rest in a given position.*

Let l be the length of the beam, W its weight and G its centre of gravity. Let $A\,G = a$ and $A\,C = b$. Let P be the horizontal force required. Let R denote the resistance of the plane at A, and Q the resistance of the wall at B.



Resolve W acting at G into two parallel forces acting at A and B; these will be $W\dfrac{l-a}{l}$ and $W\dfrac{a}{l}$ respectively, (Art. 33.) We have then one set of forces acting at A, and another set acting at B. Since there is equilibrium, the resultants of each will be equal and opposite, and therefore must act along the line $A\,B$. Let X be the value of either resultant. Then, Art. 86, at A we have three forces in equilibrium, viz., P acting horizontally, $R-W\dfrac{l-a}{l}$ acting vertically, and X acting along $A\,B$. The sides of the triangle $A\,B\,C$ are severally parallel to the directions of these three forces, and therefore, Art. 14,

$$P : R-W\frac{l-a}{l} :: A\,C : B\,C \qquad \text{i.}$$

$$\text{and} \quad P : X :: A\,C : A\,B \qquad \text{(ii.)}$$

Similarly at B we have three forces in equilibrium, viz., Q acting horizontally, $W\dfrac{a}{l}$ acting vertically, and X acting along the beam, therefore

$$Q : X :: A\,C : A\,B \qquad \text{(iii)}$$

$$Q : W\frac{a}{l} :: A\,C : B\,C \qquad \text{(iv.)}$$

From ii and iii it follows that $P = Q$, or the horizontal force to be applied at A equals the pressure against the wall. Substituting P for Q in iv, we get

$$P = W\frac{a}{l}\cdot\frac{A\,C}{B\,C} = W\frac{a\,b}{l\sqrt{l^2 - b_2}}.$$

From i and iv it follows that

$$R - W\,\frac{l-a}{l} = W\,\frac{a}{l}$$

$$\therefore \qquad R = W$$

that is, the pressure on the horizontal plane equals the weight of the beam.

*93. *One extremity of a beam is fastened by a pivot and the other rests upon a smooth horizontal plane, required the upward vertical pressure upon the pivot when the foot of the beam is acted on by a given horizontal force; the weight of the beam itself being neglected.*

Let the beam $A\,B$ (see fig., Art. 92) be fastened by a pivot at B, and let a force P be applied horizontally at A.

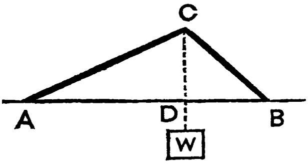
The beam is in equilibrium from the force P, the resistance of the plane acting vertically at the point A and the resistance of the pivot. Hence the resultant of P and the resistance of the plane must pass along $A\,B$. Let X denote this resultant. Then,

$$X : P :: A\,B : A\,C. \quad \therefore \quad X = P\,\frac{A\,B}{A\,C}$$

X is the pressure upon the pivot, the effect of this in the direction $C B$ is the upward vertical pressure required, this Art. 18, is equal to $X \frac{B C}{A B}$, therefore

$$\text{vertical pressure on } B = P \frac{AB}{AC} \cdot \frac{BC}{AB} = P \frac{BC}{AC}$$

*94. *Let $A C$, $B C$ be two beams of uniform thickness and density, resting upon a horizontal plane at A and B, and fastened together at C by a pin; let A and B be connected by a cord, required the tension in the cord when a weight w hangs from C.*



Let $p =$ the weight of the beam $A C$, and $q =$ the weight of the beam $B C$. Then since the beams are of uniform thickness and density, we may resolve p acting at the centre of gravity of $A C$ into two forces, each equal to $\frac{1}{2} p$, acting vertically downwards at A and C; and similarly q may be resolved into two forces, each equal to $\frac{1}{2} q$, acting vertically downwards at B and C. Hence, the whole force acting downwards at C is $w + \frac{1}{2}(p + q)$. Let $W = w + \frac{1}{2}(p + q)$. Since there is equilibrium, the upward vertical pressures upon C must equal W. Let $T =$ the tension in the cord; let $C D = h$, $A D = a$, and $B D = b$.

By the preceding article, the vertical thrust of the beam AC upon C is $T\frac{h}{a}$, and the vertical thrust of the beam BC is $T\frac{h}{b}$. Therefore

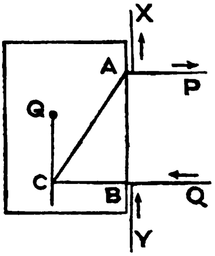
$$T\frac{h}{a}+T\frac{h}{b}=W$$

$$\therefore \qquad T=W\frac{1}{\frac{h}{a}+\frac{h}{b}}$$

If A and B are connected by a beam instead of a cord, the beam is called the *tie beam*, and T will in that case denote the horizontal thrust.

*95. *To find the horizontal strain upon the hinges of a door.*

Let G be the centre of gravity of the door, and W its weight. Let the hinges be at A and B.



Instead of the hinges, let the door be regarded as supported by a vertical force X, and a horizontal force P acting at A, and a vertical force Y and a horizontal force Q acting at B. It is evident that P and Q must act in opposite directions.

Let $G\,C$ be the vertical line drawn through G. Draw $B\,C$ perpendicular to $G\,C$. Let $B\,C = a$, and $A\,B = b$.

Remove the point of application of Y from B to A, of Q from B to C, and of W from G to C.

The resultant of W and Q at C, must be equal and opposite to the resultant of P and $X + Y$ at A. Each resultant must therefore act along $A\,C$. Let R represent the value of either; then, Art. 14, the following relations must subsist between the forces at C,

$$Q : W :: B\,C : A\,B$$

$$:: a : b \qquad \therefore \quad Q = W\frac{a}{b}$$

and $$Q : R :: B\,C : A\,C.$$

Also, since the forces at A are in equilibrium,

$$P : R :: B\,C : A\,C,$$

and $\therefore$ $$P = Q.$$

The horizontal strain upon each hinge is therefore the same in amount, while opposite in direction.

EXAMPLES.

1. A beam of uniform thickness and density, whose length is l, and weight W is placed with one end upon a horizontal plane, and the other upon the summit of an inclined plane, whose height is h and base b, the horizontal force to be applied at the foot of the beam to keep it at rest

$$= \frac{W}{2} \cdot \frac{h\sqrt{l^2 - h^2}}{b\sqrt{l^2 - h^2} + h^2}.$$

2. A triangular plate $A\ B\ C$, having the angle $A\ B\ C$ a right angle, is fastened by a pivot at C, and has weights P and Q suspended from A and B; shew that in the position of equilibrium if $A\ B = a$, and $B\ C = b$, and if x and y be the distances severally of A and B below the horizontal line drawn through C

$$x = \frac{a_2 P}{\sqrt{a^2 P^2 + b^2 Q^2}} \text{ and } y = \frac{b^2 Q}{\sqrt{a^2 P^2 + b^2 Q^2}}.$$

3. If in the preceding $A\ B\ C$ be an equilateral triangle whose side is a, prove that in the position of equilibrium

$$x = \frac{a}{2} \cdot \frac{2P + Q}{\sqrt{P^2 + Q^2 + PQ}} \text{ and } y = \frac{a}{2} \cdot \frac{2Q + P}{\sqrt{P^2 + Q^2 + PQ}}.$$

4. A carriage wheel whose weight is W and radius r rests upon a level road, shew that the horizontal force necessary to draw the wheel over an obstacle whose height is c, equals

$$\frac{W\sqrt{2ac - c^2}}{a - c}.$$

DYNAMICS.

CHAPTER I.

DEFINITIONS AND THE LAWS OF MOTION.

96. When forces not in equilibrium act upon a body, motion must ensue. The degree of rapidity with which the body moves is termed its velocity.

97. Velocity may be either uniform or variable. Velocity is uniform when the body describes *all* equal spaces in equal times, and may therefore be measured by the space described in a unit of time. It is customary to take a second for the unit of time, and a foot for the unit of measure. Thus, if a body moving with a uniform velocity have passed through 1000 feet in 8 seconds, its velocity is $\frac{1000}{8}$ or 125; and generally, if s be the space in feet through which a body has passed in t seconds, then if v denote the velocity

$$v = \frac{s}{t} \quad \text{or} \quad s = v\,t.$$

Velocity is variable when *all* equal spaces are *not* described in equal times. When a body in motion has a variable velocity, its velocity at any moment may be measured by the space which would be described in a unit of time, if at that moment the velocity were to cease to vary.

98. *The first law of motion.* A body in motion, not acted on by any external force, will continue to move in a straight line and with a uniform velocity.

This is equivalent to the assertion, that matter possesses no inherent power of changing the direction or the state of its motion.

The truth of this law must be decided by an appeal to experiment. The powers with which matter has been endowed can evidently be learnt only by observation. Every attempt to prove their existence by a-priori demonstration will be found to assume in some way or other the property under consideration.

The first law of motion cannot, however, be established by any *direct* experiment, for the prescribed conditions can under no circumstances be fulfilled. No where can we find a body which is not acted on by some external force. Every particle of matter is subject to a variety of external influences. Thus, if a ball be rolled along the ground, it is acted on by the attraction of all surrounding matter, by the resistance of the atmosphere, and by the force of friction; and it moves neither in a straight line nor with a uniform velocity. It is found, however, that the more we lessen the influence of external force, the more nearly does the motion become direct and uniform.

If a ball be rolled along a smooth pavement, it will move for a longer time, and in a line more nearly straight than when thrown with the same velocity along a rough road; and still more so if rolled along a sheet of ice.

If a weight suspended by a thread from any point be made to oscillate, it will after a time come to rest. One of the external forces acting upon the body is the resistance of the air. If this be diminished by causing the body to move within the exhausted receiver of an air pump; the oscillation will continue for a longer period;—and the more perfect the vacuum the longer will the motion continue.

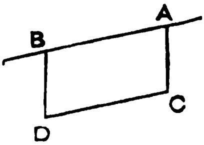
On a railway, after a train has acquired the desired velocity, it is no longer necessary for the engine to work, except so far as to overcome the effects of external forces, such as the friction of the rails and the resistance of the atmosphere.

The most convincing evidence of the truth of this law is found in the accordance of the consequences deducible from it with observed phenomena. It is impossible to doubt the correctness of a principle, upon the assumption of which the motions of the moon can be predicted with almost unerring certainty, and the time of an eclipse foretold within the fraction of a second.

99. It follows from the first law of motion, that if a force continue to act upon a body already in motion, its velocity will continually change. The magnitude of this change may be conveniently taken as a measure of the magnitude of the force.

Force thus considered is, for the sake of distinction, termed *accelerating force;* and that too, whether its effect be to accelerate or to retard the motion of the body.

100. *The second law of motion.* When a force acts upon a body in motion, the change of motion produced is the same, both in magnitude and direction, as if the force acted on the body at rest.



Thus if a body move along the line *A B*, with such a velocity, that it would describe the space *A B* in one second; and if when it arrives at *A*, a force act upon it such as of itself to cause a body to pass from *A* to *C* in one second, then at the end of the second the body will be found at *D;* the change of motion represented by *B D* being the same in magnitude and direction as if the force had acted upon the body when at rest. Each force produces its full effect in its own direction.

This law is proved by such experiments as the following:—

If a stone be dropped from the top of the mast, when a vessel is moving uniformly in any direction, and with any velocity, it will fall at the foot of the mast, just as it would if the vessel had been at rest.

If from any point a ball be let fall, and another ball be at the same instant projected forward horizontally with any velocity whatever, both balls will strike the ground at the same time. Here the ball at rest and the ball in motion are acted upon by the same vertical force, namely, the force of gravity, and both are caused to pass through the same vertical space in the same time.

If a person in an open railway carriage throw a ball perpendicularly upwards, it will not fall in rear of the train, but will drop into the hands of the individual who projected it.

101. *The third law of motion.* When pressure communicates motion to a body, the accelerating force varies as the ratio of the pressure and mass.

Thus, when a pressure P communicates motion to a mass A, and a pressure Q to a mass B, if f denote the accelerating force in the former case, and f' in the latter

$$f : f' :: \frac{P}{A} : \frac{Q}{B}.$$

If two unequal weights P and Q be attached to the extremities of a string passing over a fixed pulley, the greater weight will of course descend. The pressure, which communicates motion is the difference of the weights or $P—Q$. The mass moved is, neglecting friction, the mass of P and Q, but since at the same place, mass varies as the weight, this mass may be represented by the weight $P + Q$. It is found by experiment that if the pressure be increased while the mass remains the same, the velocity acquired after a given interval, and consequently the accelerating force, is increased in the same proportion. If the pressure and the mass be both increased in the same ratio, so that $\frac{\text{pressure}}{\text{mass}}$ remains the same, the velocity is unaffected. If, the pressure remaining the same, the mass be increased so that $\frac{\text{pressure}}{\text{mass}}$. is less than before, the velocity acquired is found to be diminished in the same proportion.

102. Since $f : f' :: \frac{P}{A} : \frac{Q}{B}$

it follows that $fA : f'B :: P : Q$.

The product of the accelerating force and mass is termed *moving force.* Hence the third law of motion may be enunciated as follows:—When pressure communicates motion to a body, the moving force is proportional to the pressure.

103. Let a pressure P communicate motion to a body whose weight is W; let f = the accelerating force, and g = the force of gravity. If W falls freely under the action of gravity, the pressure communicating motion is the weight itself; and therefore by the third law of motion

$$f : g :: \frac{P}{\text{mass}} : \frac{W}{\text{mass}}$$

but the mass moved is the same in both cases; therefore

$$f : g :: P : W$$

Hence when pressure communicates motion to a body

$$\text{accelerating force} = g \frac{\text{pressure}}{\text{weight moved}}.$$

CHAPTER II.

ON UNIFORM ACCELERATING FORCES AND GRAVITY.

104. When in consequence of the action of any force, the velocity of a body is uniformly increased, the force is termed a uniform accelerating force. Then if f represent the velocity generated in one second, $2f$ will be the velocity generated in two seconds, $3f$ in three seconds, and generally ft will represent the velocity generated in t seconds; whence if v denote this velocity

$$v = f\,t.$$

*105. *To find the space described in t seconds, when a body moves from rest under the action of a uniform accelerating force.*

Let f denote the force, and s the space described. Let the time to be divided into n equal parts.

At the end of the period $\frac{t}{n}$ the velocity will be $f\frac{t}{n}$

.... $\frac{2t}{n}$ .... $f\frac{2t}{n}$

.... $\frac{3t}{n}$ .... $f\frac{3t}{n}$.

.... $\frac{\overline{n-1}\,,t}{n}$ .... $f\frac{\overline{n-1}\,.\,t}{n}$

.... $\frac{nt}{n}$ .... ft

If during these intervals the body be supposed to move uniformly with the velocity acquired at the end of each, then (since space = vel. × time, when velocity is uniform) the space described in the first interval will be $f\,\frac{t^2}{n^2}$ in the second $f\,\frac{2\,t^2}{n^2}$ and so on; the sum of this series will be the whole space described. Therefore

$$s = f\,\frac{t^2}{n^2} + f\,\frac{2\,t^2}{n^2}. + \ldots\ldots + f\,\frac{\overline{n-1}\,.\,t^2}{n^2} + f\frac{n\ t^2}{n^2}$$

$$= f\,\frac{t^2}{n^2}\Big(1 + 2 + 3 \ldots\ldots + \overline{n-1} + n\Big)$$

$$= f\,\frac{t^2}{n^2}.\frac{n\,.\,\overline{n+1}}{2}.$$

$$= f\,\frac{t^2}{2} + f\,\frac{t^2}{2n}.$$

The error arising from supposing the velocity to be uniform through each of these intervals will diminish, if the length of the intervals be diminished, that is, if n increase, and the more n increases, the more nearly shall we approach to the actual space described. As n increases, the value of $f\,\frac{t_2}{2n}$ diminishes without limit, and ∴

$$s = \tfrac{1}{2} f\,t^2.$$

106. Gravity is found by experiment to be a uniform accelerating force. Let it be represented by g, then we

have the following relations between the velocity acquired, the space described, and the time.

$$v = g\,t.$$

$$s = \tfrac{1}{2}\,g\,t^2.$$

And hence $$v^2 = 2\,g\,s \quad \text{or} \quad v = \sqrt{2\,g\,s}.$$

The value of g as determined by experiment, is 32·2 feet.

107. Hence a body falling from rest under the action of gravity

in 1 second will fall through $\frac{g}{2}$ or 16·1 feet.
2 seconds .... $4.\frac{g}{2}$ ... $4\times16{\cdot}1$,,
3 ,, ... $9.\frac{g}{2}$ ... $9\times16{\cdot}1$,,
4 ,, ... $16.\frac{g}{2}$ ... $16\times16{\cdot}1$,,

and ∴

during 1st second a body will fall through $\frac{g}{2}$ or 16·1 ft.
2nd .... $3\frac{g}{2}$ ... $3\times16{\cdot}1$,,
3rd .... $5\frac{g}{2}$ ... $5\times16{\cdot}1$,,
4th .... $7\frac{g}{2}$ ... $7\times16{\cdot}1$,,

108. *If a body be projected with a given velocity V, and be acted on in the same direction by a uniform accelerating force f, to find the velocity acquired and the space described in a given time t.*

From the second law of motion, it follows that the velocity will equal the velocity of projection + the velocity generated by the force; and, therefore, if v denote the velocity required,

$$v = V + f t.$$

And similarly, the space described = the space due to the velocity of projection + the space due to the action of the force. The space described by a uniform velocity V in t seconds = $V t$, and the space due to the action of a uniform accelerating force in t seconds $= \frac{f t^2}{2}$. Therefore

$$s = V t + \tfrac{1}{2} f t^2.$$

109. If a body be projected with a given velocity V in a direction opposite to that in which the accelerating force acts, the expressions deduced as in the preceding article, will be

$$\begin{aligned} v &= V - f t \\ s &= V t - \tfrac{1}{2} f t. \end{aligned}$$

Squaring both sides of the first equation we have

$$\begin{aligned} v^2 &= V^2 - 2 V f t + f^2 t^2 \\ &= V^2 - 2 f (V t - \tfrac{1}{2} f t^2) \\ &= V^2 - 2 f s. \end{aligned}$$

110. *To find the height to which a body will rise when projected vertically with a given velocity V.*

Here the direction of V is opposite to that of the force of gravity or g, and hence if v denote the velocity acquired when the body has described the space s, by the preceding article,

$$v^2 = V^2 - 2 g s.$$

But when the body has attained the highest point, its velocity at that instant will $= 0$, and therefore if $s =$ the height of this point

$$0 = V^2 - 2\,g\,s.$$

$$\therefore \qquad 2\,g\,s = V^2$$

$$s = \frac{V^2}{2g}.$$

By comparing this result with the expression $v^2 = 2\,g\,s$ in Art. 106, we see that height to which the body will rise is the same as the distance through which it must fall to acquire the velocity of projection.

111. *To find the time during which a body will rise when projected vertically with a given velocity V.*

Since at the instant of attaining the greatest height $v = 0$, then. Art. 109,

$$V - g\,t = 0$$

$$\therefore \qquad g\,t = V$$

$$\therefore \qquad t = \frac{V}{g}.$$

It follows from these results that the body on its return to the point whence it was projected, will have a velocity equal to the velocity of projection, and that the times of descent and ascent will be equal.

112. Ex. 1. A stone falls from rest by the action of gravity, find the space described in 5 seconds,

$$s = \frac{g\,t^2}{2}.$$

Therefore when $t = 5$

$$s = 16 \cdot 1 \times 25 = 402 \cdot 5 \text{ feet.}$$

Ex. 2. Find the velocity which a stone will have acquired when it has fallen through 1610 feet,

$$v^2 = 2\,g\,s.$$

Therefore when $s = 1610$

$$v^2 = 2 \times 32 \cdot 2 \times 1610 = 103684$$

$$\therefore \quad v = \sqrt{103684} = 322 \text{ feet per second.}$$

Ex. 3. A body is projected vertically upwards with a velocity of 100 feet per second, to find how high it will rise.

By Art. 110, $s = \frac{V^2}{2g}$, therefore in this case

$$s = \frac{10000}{64 \cdot 4} = 155 \cdot 28.$$

Ex. 4. A body is projected vertically upwards with a velocity of 161 feet per second, to find how long it will continue to ascend.

By Art. 111, $t = \frac{V}{g}$, therefore in this case

$$t = \frac{161}{32 \cdot 2} = 5 \text{ seconds.}$$

Ex. 5. A stone is projected vertically upwards and returns to the same spot after an interval of 12 seconds, find the velocity of projection, and the height to which the body has risen.

The velocity of projection is equal to the velocity acquired by the body during the time of descent, or 6 seconds; and therefore if V be the velocity of projection, since $v = g\,t$, Art. 106.

$$V = 32{\cdot}2 \times 6 = 213{\cdot}2 \text{ feet per second.}$$

The height to which the body has risen is equal to the distance through which it falls from rest during 6 seconds, that is,

$$\text{the height required} = \frac{g}{2}\,6^2$$

$$= 16{\cdot}1 \times 36 = 579{\cdot}6 \text{ feet.}$$

*113. *To find the force which accelerates a body down an inclined plane.*

Let a weight W rest upon an inclined plane, whose length $= l$, and height $= h$. W would be kept at rest on this plane by force acting along the plane $= W\frac{h}{l}$, Art. 76. Therefore $W\frac{h}{l}$ is the pressure which causes the body to move down the plane. But when pressure communicates motion to a body, accel. force $= g \,.\, \frac{\text{pressure}}{\text{weight moved}}$. Art. 103. Therefore in this case

$$\text{the accelerating force} = g \,.\, \frac{W\frac{h}{l}}{W}$$

$$= g\,\frac{h}{l}.$$

*114. ***Required the time of descent when a body falls down an inclined plane.***

By the general formula, Art. 105, $s = \frac{ft^2}{2}$. Let h = height of the plane, and l = its length. Then in this case $s = l$ and by the preceding article $f = g \cdot \frac{h}{l}$, and therefore

$$l = \frac{gh}{l} \cdot \frac{t^2}{2}$$

$$\therefore \quad t^2 = \frac{2l^2}{gh}$$

$$\therefore \quad t = \sqrt{\frac{2l^2}{gh}}$$

$$= l\sqrt{\frac{2}{gh}}.$$

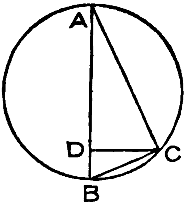
*115. ***To find the velocity acquired by a body falling down an inclined plane.***

By the general formula $v = \sqrt{2fs}$. As in the preceding article $s = l$, and $f = g\frac{h}{l}$. Therefore

$$v = \sqrt{2gh}.$$

But $\sqrt{2gh}$ expresses the velocity which a body acquires in falling freely under the action of gravity, through a distance h; therefore the velocity acquired by a body falling down an inclined plane, is the same as that acquired in falling through the perpendicular height. Consequently, whatever be the length of the plane, the velocity acquired is the same if the height remains the same.

*116. *If a circle be placed with its plane vertical, the times of descent down all chords drawn through its highest or lowest points are equal.*



Let $A\ B\ C$ be any circle whose plane is vertical. Let $A\ B$ be the vertical diameter, and $A\ C$ any chord drawn through A. Let t be the time in which a body falls down $A\ C$, then Art. 114.

$$t = \sqrt{\frac{2\ A\ C^2}{g.\ A\ D}}$$

The triangles $A\ C\ D$, $A\ C\ B$ are similar, therefore

$$A\ D\ :\ A\ C\ ::\ A\ C\ :\ A\ B, \text{ or } \frac{A\ C^2}{A\ D} = A\ B$$

Hence $$t = \sqrt{\frac{2.\ A\ B}{g}}.$$

As this result is independent of the position of the point C, it follows that the times of descent down all chords drawn through A are equal.

And similarly, if $B\,C$ be any chord drawn through B, the time of descent is

$$\sqrt{\frac{2.\,B\,C^2}{g.\,B\,D}}$$

but $\frac{B\,C^2}{B\,D} = A\,B$, therefore as before the time of descent is

$$\sqrt{\frac{2.\,A\,B}{g}}.$$

*117. *To find the accelerating force when a weight P draws another weight W along an inclined plane, (the string connecting them passing along the plane); and the time in which P will draw W along the whole length of the plane.*

If P were equal to $W\frac{h}{l}$, P would just support W, (Art. 76), therefore the pressure causing motion is $P - W\frac{h}{l}$, the weight of the mass moved is $P + W$, and therefore if f denote the accelerating force

$$f = g\,.\frac{P - W\frac{h}{l}}{P + W}$$

$$= g\,.\frac{P\,l - W\,h}{l\,(P + W)}$$

Let t = the time in which P will draw W along the whole length of the plane. In the general formula

$s = \frac{ft^2}{2}$ for s substitute l, the length of the plane, for f the value just determined, then

$$l = g \cdot \frac{P\,l - W\,h}{l\,(P + W)} \cdot \frac{t^2}{2}$$

$$\therefore \qquad t = l \cdot \sqrt{\frac{2}{g} \cdot \frac{P + W}{P\,l - W\,h}}$$

*118. Ex. A body of 9 ounces draws a body of 7 ounces over a fixed pulley, find the space described from rest in t seconds, neglecting the inertia of the pulley.

The pressure causing motion is $9 - 7 = 2$ ounces, and the weight moved is $9 + 7 = 16$ ounces, therefore

$$\text{the accelerating force} = g \cdot \frac{2}{16} = 4 \text{ nearly}$$

Substituting this in the general formula $s = \frac{ft^2}{2}$, we have

$$\text{space described} = \frac{4 \cdot t^2}{2} = 2\,t^2.$$

Hence, is one second the space described is 2 feet, in two seconds 8 feet, in three seconds 18 feet, and so on.

*119. A weight of 9lbs. is drawn along a smooth horizontal table by a weight of 1lb. hanging vertically by a string passing over a pulley at the edge of the table; find the space described from rest in 3 seconds, the velocity acquired in 4 seconds, and the space described in the 5th second.

The pressure causing motion is 1lb., the weight moved is $9 + 1 = 10$lbs. Therefore

$$\text{The accelerating force,} = g.\frac{1}{10} = 3{\cdot}2.$$

$$\text{The space described in 3 seconds} = \frac{3{\cdot}2}{2}.\,9 = 14{\cdot}4 \quad \text{feet.}$$

$$\text{The velocity acquired in 4 seconds} = 3{\cdot}2 \times 4 = 12{\cdot}8$$

The space described in the 5th second, equals the space described in 5 seconds — the space described in 4 seconds, therefore

$$\begin{aligned}\text{space described in the 5th second} &= 1{\cdot}6 \times 25 - 1{\cdot}6 \times 16\\ &= 1{\cdot}6 \times 9\\ &= 14{\cdot}4 \text{ feet.}\end{aligned}$$

HYDROSTATICS.

CHAPTER I.

FUNDAMENTAL PROPERTIES OF FLUIDS.

120. Hydrostatics is the science which treats of the equilibrium of forces acting upon fluid bodies.

121. A fluid is a body, all of whose parts can move freely amongst themselves.

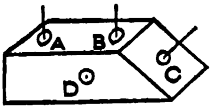
Motion consequently can be caused amongst the particles of a fluid body, by the application of the slightest conceivable force.

122. Fluids are divided into liquids and aeriform fluids. An aeriform fluid is distinguished from a liquid by the existence of an expansive or repulsive force amongst its particles, in consequence of which they tend in a greater or less degree to move off from one another.

123. When a fluid is at rest, any portion of it may be supposed to become solid, without disturbing the equilibrium.

Since all the particles of the fluid are at rest, it is clear that they will not less be so, if any number of them be rigidly connected with each other, and so deprived of the power of relative motion.

124. *When pressure is communicated to a fluid mass in equilibrium, it is transmitted equally and in all directions.*



In the sides of a closed vessel of any shape, let a number of apertures of equal area be made, and let these apertures be supplied with pistons exactly fitting them. Let the vessel be filled with water, and let the pistons be maintained by some mechanical contrivance in their respective positions.* If any one of the pistons be pressed in with a force P, it is found that all the other pistons experience a similar pressure, and that a corresponding force P must be applied to each of them, in order to retain them in their places. Thus the pressure communicated to the fluid has been transmitted in all directions, for all the pistons experience it, whatever their position and whatever the shape of the vessel; and the transmitted pressure is equal to that communicated.

These results are the same, whatever the number of the pistons and wherever placed. It follows, therefore, that every

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* For the reason of this provision, see Art. 187.

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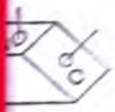

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pistons working in water tight collars *c c, d d.* The piston *B*, the diameter of which is much greater than that of *A*, supports a plate *E*, upon which the substance to be pressed is placed. *A* is capable of being worked up and down by a lever *G H* having its fulcrum at *G*. L is a pipe leading into a reservoir *I*, *a* is a valve opening upwards, and *b* a valve opening into the vessel *D*.

Let the vessels *C, D* be supposed to be filled up with water when the pistons are in the position represented in the figure. Let *A* be forced down with a pressure *P*, then by the preceding article *B* is forced up with a pressure $= P \times \frac{\text{area of } B}{\text{area of } A}$.

If *B* yield to this pressure, the piston *A* will descend and a portion of the water in *C* will be forced into the vessel *D*. The return of the fluid will be prevented by the valve *b*. If the piston *A* be now raised, fresh water will be pumped up into the vessel *C* from the reservoir *I*, and if *A* be again forced down with a pressure *P*, *B* will be again forced up with the same pressure as before. This process may be continued as long as the substance yields to the pressure existed upon it.

The pressure upon *B* may at any time be removed by unscrewing the plug *e*, by which the water is allowed to flow back into the reservoir.

127. The safety valve is another important application of the same principle. A portion of the boiler of a steam engine whose area contains *a* square inches, is furnished with a valve opening outwards. The valve is so contrived as to admit of

being loaded with certain weights. If a weight of P lbs. be placed on the valve, it is pressed down with a pressure of $\frac{P}{a}$ lbs. to the square inch. If then at any time the pressure of the steam be greater than $\frac{P}{a}$ lbs. to the square inch, the valve will be forced open, and the steam will escape; and since the pressure of the steam is the same on every inch of the surface of the containing vessel, no portion of the boiler will experience a pressure greater than $\frac{P}{a}$ lbs. to the square inch. So that if $\frac{P}{a}$. be less than the pressure per square inch, which the weakest portion of the boiler can bear, the boiler can never burst.

CHAPTER II.

ON THE EQUILIBRIUM OF FLUIDS ACTED UPON BY GRAVITY.

128. All the particles of a fluid equally with those of any solid body, are subject to the action of gravity. Any portion, therefore, however small of any fluid, is drawn towards the earth with a certain degree of force, that is, is possessed of weight. Other forces besides that of gravity may act upon the particles of a fluid, and when this is the case, the determination of the conditions of equilibrium is difficult. We here confine ourselves to an examination of the circumstances which attend the equilibrium of fluids when their particles are acted upon by the force of gravity alone.

129. *The surface of a fluid at rest is horizontal.*

For, if possible, let any portion of the surface of a fluid at rest not be horizontal. There can then be taken in the surface, two points which do not lie in the same horizontal line. Let the fluid be supposed to be divided by a plane passing through these two points, and let the portion of fluid below and that above be supposed to become solid. The latter will then be a weight unsupported by any force at rest upon an inclined plane. But since there is no friction between the particles of a fluid this is impossible. Therefore no two points can be taken in the surface of a fluid at rest, which do not lie in the same horizontal line, and consequently the surface of a fluid at rest must be horizontal.

130. **The pressure of a fluid at rest upon any plane, must be in a direction perpendicular to the plane.**

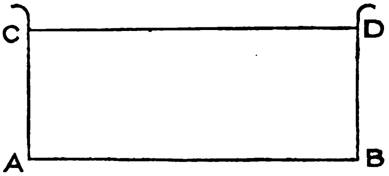
For the only force opposed to the pressure of the fluid on the plane, is the resistance of the plane itself, and since there is equilibrium, these two forces must be equal and opposite. The resistance of the plane is perpendicular to the plane; this consequently must be the direction of the pressure of the fluid.

131. The total pressure of a fluid at rest upon any surface may be, and most commonly is, the sum of two different pressures, the one a pressure transmitted simply by the fluid, and the other a pressure arising from the weight of the fluid itself. The former, as we have seen, is the same upon every unit of a surface in contact with the fluid. The latter, it will be seen hereafter, varies with the position of the surface. When the latter is known in any case, it is only necessary to add to it the amount of transmitted pressure corresponding to the area of the surface, and the total pressure is found. If then we can determine the pressure of a fluid upon any surface arising from the action of gravity upon its particles, upon the supposition that no external pressure is communicated to the fluid, the problem is completely solved.

132. *To determine the pressure of a fluid upon the upper surface of a plane immersed in it, when the plane is parallel to the surface of the fluid.*

The given plane may be regarded as forming part of the vessel, for the fluid beneath a plane can evidently exert no pressure upon its upper surface.

First:—Let the sides of the containing vessel be vertical.



In this case it is clear that the pressure upon the bottom of the vessel must be equal to the weight of the fluid. Let w be the weight of a cubic unit of the fluid; let S be the area of the bottom of the vessel and h the height of the fluid. The total pressure upon the bottom of the vessel is, therefore,

$$w\,S\,h$$

This pressure is equally distributed over the whole area, for there can be no reason why it should be greater at one point than at another, therefore

$$\text{pressure upon each unit of area} = \frac{w\,S\,h}{S} = w\,h.$$

Hence if a be the area of the given plane,

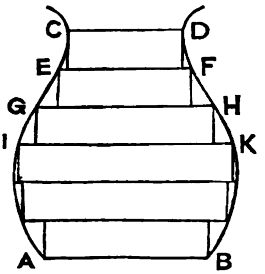
$$\text{pressure upon the given plane} = w\,a\,h.$$

If a force act upon $C\,D$ the surface of the fluid, so as to exert a pressure p upon a unit of area. Since this pressure is transmitted equally and in all directions, the pressure upon each unit of area in the bottom of the vessel will be

$$w\,h + p.$$

Secondly:—Let the vessel be of any shape whatever.

Let the fluid be divided by horizontal planes into a number of thin plates. When the number of plates is large, we may without sensible error suppose the sides of each to be vertical.



Let h be the weight of the fluid, and z the thickness of each plate, so that $z = \frac{h}{n}$ where n is some large number. The pressure on the lower surface of each plate equals the pressure arising from the weight of the fluid in each plate, together with the pressure exerted upon its upper surface by the fluid above it.

By the former part of this article,

pressure on each unit of area in $EF = w.z$.

And,

pressure on each unit of area in $GH = w.z$ + pressure on EF

$$= w.z + w.z$$
$$= w.2z.$$

Also,

pressure on each unit of area in $IK = wz$ + pressure on GH

$$= wz + w.2z$$
$$= w.3z,$$

and so for the rest, therefore,

$$\begin{aligned}\text{pressure on each unit of area in } A\,B &= w.\,n\,z\\ &= w.\,h.\end{aligned}$$

And if as before a be the area of the given plane, the total pressure upon its upper surface is

$$w.\,a\,h.$$

Any plane parallel to the surface of the fluid may be regarded either as the lower surface of the portion of fluid above it, or as the upper surface of the portion of fluid below it. And since the fluid is supposed to be in equilibrium, the downward pressure upon the plane exerted by the fluid above it, must be counterbalanced by the upward pressure exerted by the fluid below. Therefore in every case;

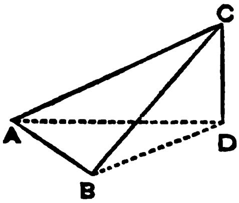
The pressure of a fluid upon any plane parallel to its surface is equal to the weight of a column of the fluid, whose base is equal to the area of the plane, and whose height is equal to the depth of the plane below the surface of the fluid.

133. *To determine the pressure of a fluid upon a triangular plane inclined to the horizon at any angle, but having one of its sides parallel to the surface of the fluid.*

First.—When the plane is a right angled triangle.

Let $A\ B\ C$ be any right angled triangle, having the angle at B a right angle, and the side $A\ B$ parallel to the surface of the fluid.

Let $A\ B\ D$ be the horizontal plane drawn through $A\ B$. Let the vertical planes drawn through $A\ C$ and $B\ C$ intersect the horizontal plane in $A\ D$ and $B\ D$ respectively, and each other in $C\ D$. Then will $C\ D$ be perpendicular to the horizontal plane, and the angle $C\ B\ D$ will be the angle at which the given plane is inclined to the horizon.



Let the portion of fluid included in the pyramidal figure $A\ B\ C\ D$ be conceived to become solid. Since the equilibrium is not thereby disturbed, the vertical forces acting upon the body $A\ B\ C\ D$ must be in equilibrium. The forces acting upon the pyramid are its weight and the pressures upon the sides. Since the sides $A\ C\ D$, $B\ C\ D$ are vertical, the pressures upon them exert no influence in the vertical direction, Art. 27. The vertical forces are consequently the weight of the pyramid, and the vertical component of the pressure upon $A\ B\ C$ which act downwards, and the pressure upon the horizontal plane $A\ B\ D$ which acts upwards.

Let P be the pressure upon $A\ B\ C$. Let $A\ B = a$, $B\ C = b$, $B\ D = c$, and $C\ D = h$. Let z be the distance of C below the surface of the fluid, and w the weight of a cubic unit of fluid. Then

$P\frac{c}{b} =$ vertical component of P, (Art. 79)

$w.\frac{ac}{2}.\frac{h}{3} =$ weight of $A\ B\ C\ D$*

and $w.\frac{ac}{2}(z + h) =$ pressure upon $A\ B\ D$.

Therefore,

$$P\frac{c}{b} + w.\frac{ac}{2}\cdot\frac{h}{3} = w\frac{ac}{2}(z + h)$$

$$\therefore \qquad P\frac{c}{b} = w\frac{ac}{2}\left(z + \frac{2h}{3}\right)$$

$$\therefore \qquad P = w\frac{ab}{2}\left(z + \frac{2h}{3}\right).$$

But $z + \frac{2h}{3}$ is the distance of the centre of gravity of $A B C$ from the surface of the fluid, and $\frac{ab}{2}$ is the area of $A\ B\ C$; therefore the pressure on $A\ B\ C$ is equal to the weight of a column of fluid, whose base is equal to the area of $A B C$, and whose height is equal to the distance of the centre of gravity of $A\ B\ C$ from the surface of the fluid.

If the given plane be vertical, the construction adopted above will become impossible. The determination however of the pressure in this case easily follows from the preceding. Let the figure represent a portion of a fluid at rest. Let the base $A\ B\ F\ G$ be a rectangle whose plane is horizontal, and let the two ends *A B C*, *E F G* be right angled triangles having their right angles at *B* and *F* respectively. Let the triangle *E F G* be vertical, and let *C E* be parallel to *B F*.

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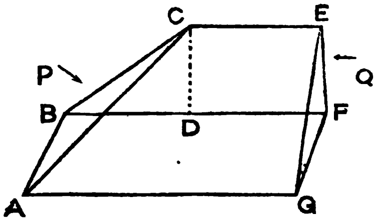
* The content of any pyramid is equal to one third of the product of its base and height



Let the portion of fluid included in the wedge-like figure *A B C E F G* become solid. The only forces tending to cause motion in the direction of *B F* or *A G* are the horizontal pressures on the ends *A B C*, *E F G*, and since the body is at rest these must be equal.

Let Q be the pressure on the vertical triangle *E F G*, and P the pressure on the triangle *A B C*. Through C draw $C D$ perpendicular to $B F$. CD will be equal to $E F$. Let $A B = a$, $B C = b$, $E F = h$, and let z be the distance of C or E below the surface of the fluid.

By Art. 79, the horizontal pressure of P is $P \dfrac{C D}{B C}$ or $P \dfrac{h}{b}$

Therefore,

$$Q = P \frac{h}{b}.$$

But according to the result obtained above, $P = w \dfrac{a b}{2}\left(z + \dfrac{2h}{3}\right)$

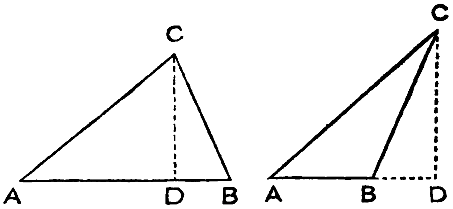
L

therefore,

$$Q = w\frac{ah}{2}\left(z + \frac{2h}{3}\right).$$

$F\,G = A\,B = a$, therefore $\frac{ah}{2}$ = area of $E\,F\,G$, and $z + \frac{2h}{3}$ is the distance of the centre of gravity of $E\,F\,G$ from the surface of the fluid.

Secondly.—When the plane is an oblique angled triangle.



Let $A\,B\,C$ be the given triangle in which $A\,B$ is parallel to the surface of the fluid. Draw $C\,D$ perpendicular to $A\,B$ or $A\,B$ produced. The given triangle will then be either the sum or the difference of the two right angled triangles $A\,C\,D$, $B\,C\,D$.

Let P be the pressure on $A\,B\,C$, X the pressure on $A\,C\,D$, and Y the pressure on $B\,C\,D$; then in the one case P will equal the sum, and in the other case the difference of X and Y.

Since the base $A\,B$ is parallel to the surface of the fluid, the centres of gravity of $A\,B\,C$, $A\,C\,D$, $B\,C\,D$ will be at the same distance from the surface, let z be this distance, let $A\,B = a$, $A\,D = x$, $B\,D = y$, and $C\,D = h$.

By the former part of this article

$$X = w\frac{xh}{2}z \text{ and } Y = w\frac{yh}{2}z.$$

Then since P is equal either to $X + Y$ or to $X - Y$

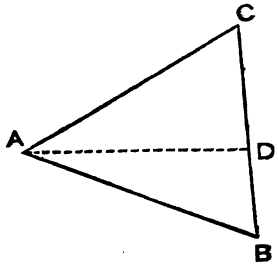
$$\begin{aligned} P &= w\frac{xh}{2}z + \frac{yh}{2}z & \text{or} &= w\frac{xh}{2}z - w\frac{yh}{2}z \\ &= w\frac{(x+y)h}{2}z & \text{,,} &= w\frac{(x-y)h}{2}z \\ &= w\frac{ah}{2}z & \text{,,} &= w\frac{ah}{2}z. \end{aligned}$$

But $\frac{ah}{2}$ = the area of $A\ B\ C$, and z is the distance of the centre of gravity of $A\ B\ C$ from the surface of the fluid. Hence the pressure of a fluid upon any triangular plane, inclined to the horizon at any angle, but having one of its sides parallel to the surface of the fluid, is equal to the weight of a column of the fluid, whose base is equal to the area of the plane, and whose height is equal to the depth of the centre of gravity of the plane below the surface of the fluid.

134. *To determine the pressure of a fluid upon any triangular plane.*

If any side of the triangle be parallel to the surface of the fluid, the pressure has been determined in the preceding article. If no side be parallel to the surface of the fluid, let A be that vertex of the triangle which is neither the highest nor the lowest.

Let $A\,D$ be parallel to the surface of the fluid, and the given triangle will be divided into two triangles, each of which has one side parallel to the surface of the fluid.



Let A be the area of the triangle $A\,B\,C$, and z the distance of its centre of gravity from the surface of the fluid. Let A_1, A_2 be the areas respectively of $A\;C\;D$ and $A\;B\;D$, and z_1, z_2 the distances of their centres of gravity from the surface of the fluid.

By Art. 133, pressure on $A\;C\;D = w\,A_1\,z_1$

and pressure on $A\;B\;D = w\,A_2\,z_2$

$\therefore$ pressure on $A\;B\;C = w\,A_1\,z_1 + w\,A_2\,z_2$

$$= w\,(A_1\,z_1 + A_2\,z_2)$$

But (Art. 51) $A\,z = A_1\,z_1 + A_2\,z_2$

$\therefore$ pressure on $A\;B\;C = w\,A\,z$,

that is, the pressure upon the triangular plane is the weight of a column of the fluid, whose base equals the area of the triangle, and whose height is equal to the depth of the centre of gravity of the triangle below the surface of the fluid.

135. *To determine the pressure of a fluid upon a plane bounded by any rectilinear figure.*

Let the plane be divided into triangles. Let A be the area of the whole plane, and z the distance of its centre of gravity below the surface of the fluid. Let A_1, A_2, A_3, &c. be the areas of the several triangles, and z_1, z_2, z_3, &c. the distances of their centres of gravity below the surface of the fluid.

$$\text{pressure on given plane} = w\,A_1\,z_1 + w\,A_2\,z_2 + w\,A_3\,z_3 + \dots$$
$$= w\,(A_1\,z_1 + A_2\,z_2 + A_3\,z_3 + \dots)$$

But (Art. 51) $A\,z = A_1 z_1 + A_2 z_2 + A_3 z_3 + \dots$

$\therefore$ pressure on given plane $= w\,A\,z$,

or the pressure on the plane is the weight of a column of the fluid whose base equals the area of the given figure, and whose height is equal to the depth of the centre of gravity of the figure below the surface of the fluid.

136. The proposition thus proved respecting all plane surfaces bounded by any rectilinear figure can be shewn to apply also to any surface whatever. The proof is not sufficiently elementary to be given here. The Student, however, may receive it as an established principle, that *the total pressure of a fluid upon any surface whatever, is equal to the weight of a column of the fluid, whose base equals the area of the given surface, and whose height is equal to the depth of the centre of gravity of the surface below the surface of the fluid.*

It follows as a corollary from this, that if the centre of gravity of the surface be fixed, the pressure of the fluid upon the surface is the same, whatever position it may be made to occupy.

137. If the Student will now revert to Art. 124, he will see why it was necessary to insert the provision that the pistons be maintained in their position by some mechanical contrivance. For if the pistons, (as is the case with the pistons C, D), be situated any where except upon the uppermost surface of the vessel, they will experience a pressure greater or less according to their distance below the surface of the fluid, and will in consequence of this pressure be forced out if not prevented.

The pistons A and B which press upon the uppermost surface of the fluid, experience no pressure from the weight of the fluid itself; whatever force therefore is impressed upon them is transmitted undiminished to every part of the fluid. But if a pressure P be applied elsewhere, at C for instance, and no other force act upon the piston, then a part of this pressure will be employed in counterbalancing the pressure of the fluid upon C, and the remainder only will be the pressure transmitted by the fluid. Hence if a be the area of any piston, h the distance of its centre of gravity below the surface of the fluid, and w the weight of a cubic unit of the fluid, and if P be the pressure exerted upon the piston, the pressure transmitted to every corresponding area is

$$P - w\,a\,h,$$

or the pressure transmitted to every unit of area is

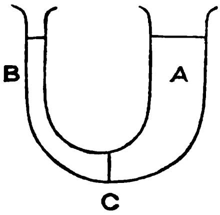
$$\frac{P}{a} - w\,h.$$

138. Since the pressure of a given fluid upon a given horizontal plane, depends only upon the depth of the plane beneath the surface of the fluid, (Art. 132,) it follows that, if the bottoms of any number of vessels of various shapes be of

equal area, and the same fluid be poured to the same height in all, the pressure upon the bottom of each vessel will be the same, however different the quantity of fluid in each may be. If the bottoms of the vessels be moveable, it will be found by experiment that the same force is necessary in each case to retain them in their respective positions.

139. *If a fluid be poured into any one of a number of open vessels having a free communication with each other, the fluid will rise to the same height in all.*

Let A and B be any two such vessels, having the lowest point at C.



Let a thin vertical plate of the fluid passing through C become rigid. Let a be the area of this plate, h the distance of its centre of gravity below the surface of the fluid in A, and h' its distance below the surface in B.

Let w be the weight of a cubic unit of the fluid. Then the pressure exerted on the one side of the plate by the fluid in A is $w\,a\,h$, and that exerted on the other side by the fluid in B is $w\,a\,h'$. But since there is equilibrium, these pressures must be equal, therefore

$$w\,a\,h = w\,a\,h',$$

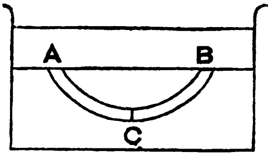
or

$$h = h'$$

that is, the surface of the fluid is at the same height in both. In like manner the proposition may be proved when more than two vessels are in free communication.

140. *The pressures of a fluid at rest, at any two points in the same horizontal plane are equal.*

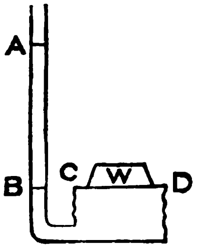
Let A and B be any two points in a fluid at rest, lying in the same horizontal plane. Let P be the pressure of the fluid upon a unit of area at A, and Q the pressure upon a unit of area at B; then shall P and Q be equal.



Let $A\ C\ B$ be any portion of the fluid detached from the rest and enclosed in a rigid tube. Let C be the lowest point of this tube, and let a vertical plate at C become rigid. The pressures on each side of this plate arising from the fluid in $A\ B\ C$ are equal, and consequently since the fluid is at rest the pressures transmitted from A and B must be equal, that is P and Q must be equal.

141. *Hydrostatic Bellows.* $A\ B$ is a narrow tube communicating with a vessel formed by uniting together two pieces of wood by some flexible and waterproof substance. If water be poured down the tube it will enter the vessel, and raise a large weight W placed upon $C\ D$.

When equilibrium exists, let A be the surface of the water in the tube, and let B lie in the same horizontal plane with $C\,D$. Let $A\,B = h$, and let w be the weight of a cubic unit of water.



The pressure upon a unit of area at B is $w\,h$, and since $C\,D$ is in the same horizontal plane with B, the pressure on each unit of area in $C\,D$ must also be $w\,h$. Therefore if $b =$ number of units in the area of $C\,D$, the total pressure on $C\,D$ $= w\,b\,h$, and since there is equilibrium this must equal the weight supported, therefore

$$W = w\,b\,h$$

If $a =$ area of the horizontal section of the pipe at B,

$$\text{weight of fluid in } A\,B = w\,a\,h$$

$$\therefore \qquad W = \frac{b}{a} \times \text{weight of fluid in } A\,B.$$

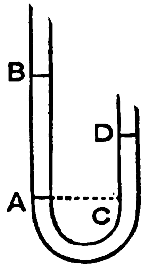
If $C\,D$ and the tube are circular, and if R be the radius of $C\,D$, and r the radius of horizontal section of the tube.

$$\frac{b}{a} = \frac{R^2}{r^2}$$

$$\therefore \qquad W = \left(\frac{R}{r}\right) \times \text{weight of fluid in } A\,B.$$

*142. *To determine the position of equilibrium when two fluids which do not mix meet in a bent tube.*

Let A be the common surface to the two fluids. Let B be the surface of the one fluid, and D that of the other when there is equilibrium.



Let h be the distance of B above the horizontal plane at A, and h' the distance of D above the same plane.

Let w be the weight of a cubic unit of the fluid in $A\,B$, and w' of the fluid in $A\,C\,D$. Let a be the area of the horizontal section of the tube at A. If a thin plate of the fluid at A be supposed to become solid, since there is equilibrium, the pressures on its upper and under surfaces must be equal. The pressure on its upper surface is $w\,a\,h$, and on its lower surface $w'\,a\,h'$, therefore

$$w\,a\,h = w'\,a\,h'$$

$$w\,h = w'\,h'$$

$$\therefore \quad \frac{h}{h'} = \frac{w'}{w}.$$

143. *To find the resultant of the pressure of a fluid on the surface of a solid, wholly or partially immersed in it.*

Let any portion M of any fluid at rest become solid, and since the equilibrium is not thereby disturbed, the forces acting upon M must be in equilibrium. The only forces acting upon M are the weight of M acting vertically downwards at the centre of gravity of M, and the pressures of the fluid upon the surface of M. Hence the resultant of these pressures must be equal and opposite to the weight of M acting at its centre of gravity. But the fluid will exert the same pressure upon the surface of any other body of the same form as M occupying its place. Hence the resultant of the pressure of a fluid on the surface of a solid immersed in it, is equal to the weight of the fluid displaced, and acts vertically upwards through the centre of gravity of the fluid displaced.

144. If any body hang freely by a cord, we have seen (Art. 41,) that the centre of gravity lies in the vertical line drawn through the point of suspension, that is in the line of the cord; and the tension in the cord equals the weight of the body.

If such a body be wholly immersed in a fluid, it has been shewn in the preceding article, that it will be pressed upwards by a force equal to the weight of the fluid displaced, which in this case is the weight of a portion of the fluid equal to the bulk of the body; and this pressure is directly opposite to the weight of the body, therefore the tension in the cord will be the weight of the body diminished by the weight of an equal bulk of the fluid.

*145. *To determine the conditions of equilibrium of a floating body.*

If a body floating in a fluid be at rest, the forces acting upon it are the weight of the body, which acts at its centre of gravity, and the resultant pressure of the fluid which acts at the centre of gravity of the fluid displaced.

The necessary conditions of equilibrium are, that these forces be *equal* and *opposite*. The two conditions therefore are,

First:—That the weight of the body be equal to that of the fluid displaced; and

Secondly:—That the centres of gravity of the body and the fluid displaced be in the same vertical line.

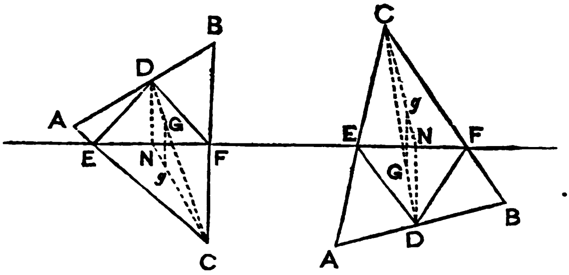
As a body cannot displace a quantity of fluid greater than its own bulk, if the weight of the body be greater than that of an equal bulk of the fluid, the body cannot float in that fluid. If the weight of the body be exactly equal to that of an equal bulk of the fluid, the body will have in no position any tendency either to rise or to sink, provided only it be entirely covered by the fluid.

*146. *Ex. If a triangular plate float vertically in any fluid, in the position of equilibrium, the lines joining the bisection of the side, which is either wholly within or wholly without the fluid, with the points at which the other sides meet the surface of the fluid; shall be equal to one another.*

Let $A\ B\ C$ be a triangular plate floating vertically in any fluid. Let the side $A\ B$ be either wholly within or wholly without the fluid, and let the other sides meet the surface of the fluid in the points E and F. Then if $A\ B$ be bisected in D, the lines $D\ E$ and $D\ F$ shall be equal.

Bisect $E\ F$ in N, and draw $C\ D$, $C\ N$ and $D\ N$.

Let $D\,G = \frac{1}{3}\,D\,C$, then, Art. 44, G is the centre of gravity of $A\ B\ C$.



Let $N\,g = \frac{1}{3}\,N\,C$, then g is the centre of gravity of $E\ F\ C$, that is of the fluid displaced.*

Join $G\ g$, then since the floating body is at rest, the line $G\ g$ must be vertical, Art. 145. But $D\ N$ is parallel to $G\ g$, (Euclid B. vi. Prop. ii.); therefore $D\ N$ is vertical, and consequently at right angles with $E\ F$, since the surface of a fluid at rest is horizontal.

Then in the two triangles $E\ N\ D$, $F\ N\ D$, the sides $E\ N$ and $F\ N$ are equal, the side $D\ N$ common, and the angles $D\ N\ E$, $D\ N\ F$ equal, being right angles; therefore $D\ E$ equals $D\ F$.

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* This applies only to the left hand figure. In the other figure, since G is the centre of gr. of the whole triangle $A\ B\ C$, and g of the triangle $C\ E\text{-}F$, the centre of gr. of $A\ B\ F\ E$ or of the fluid displaced must lie in the line $g\ G$.

CHAPTER III.

ON SPECIFIC GRAVITY.

147. The Specific Gravity of any Substance is the ratio of the weights of equal bulks of that substance, and of a certain standard substance.

The Standard with which all substances except gases are compared, is distilled water at a temperature of 60°.

The Standard for gases is pure atmospheric air at a temperature of 60°.

148. The Hydrostatic Balance is, in its simplest form, a common pair of scales with a fine thread attached to the under surface of one of the scale pans. By this arrangement, any solid substance can be weighed both in the ordinary way, and also, by means of the thread, when immersed in any fluid.

149. *To find the specific gravity of a body heavier than water.*

Let W be the weight of the body in air, and w its weight in water. It has been seen, Art. 144, that w differs from W by the weight of a quantity of water equal to the bulk of the body immersed. Hence,

weight of water equal in bulk to the given body $= W - w$.

$$\therefore \quad \text{sp. gr. of the given body} = \frac{W}{W - w}.$$

150. *To find the specific gravity of a body lighter than water.*

Let A be the given body, and B any other body, such that when attached to A both will sink in pure distilled water.

Let W be the weight of $A + B$ in air, and w their weight in water. Let W' be the weight of B in air, and w' its weight in water. Then:

weight of water equal in bulk to $A + B = W - w$

" " " $B = W' - w'$

∴ " " " $A = W - w - (W' - w')$

But weight of A in air $= W - W'$

∴ sp. gr. of $A = \dfrac{W - W'}{W - w - (W' - w')}$

151. *Giving the weights and specific gravities of the components required the specific gravity of the compound.*

Let the compound be formed by the mixture of a body A, whose weight is W and sp. gr. s with a body B whose weight is W' and sp. gr. s'. Then by the definition (Art. 147).

$s = W$: weight of water equal in bulk to A,

and $s' = W'$: weight of water equal in bulk to B.

∴ weight of water equal in bulk to $A = \dfrac{W}{s}$

and weight of water equal in bulk to $B = \dfrac{W'}{s'}$

∴ weight of water equal in bulk to the compound $\dfrac{W}{s} + \dfrac{W'}{s'}$

But weight of compound in air $= W + W'$

∴ sp. gr. of compound $= \dfrac{W + W'}{\dfrac{W}{s} + \dfrac{W'}{s'}}$

It is assumed here that the volume of the compound is the sum of the volumes of the component bodies. This, however, is not always the case, as very frequently the volume of the compound is affected by the chemical combination of the components.

152. If the bulks of the components are known and not their weights, the specific gravity of the compound may be found as follows. Let V, V' be the number of cubic units in A and B respectively; then if w be the weight of a cubic unit of A and w' of B.

$$W = V\,w \text{ and } W' = V'\,w'$$

$$\therefore \qquad \frac{W}{W'} = \frac{V\,w}{V'\,w'}$$

But

$$\frac{w}{w'} = \frac{s}{s'}.$$

$$\therefore \qquad \frac{W}{W'} = \frac{V\,s}{V'\,s'}.$$

$$\text{sp. gr. of compound} = \frac{W + W'}{\dfrac{W}{s} + \dfrac{W'}{s'}} \qquad \text{(by preceding Art.)}$$

$$= \frac{\dfrac{W}{W'} + 1}{\dfrac{W}{W's} + \dfrac{1}{s'}}$$

$$= \frac{\dfrac{V\,s}{V'\,s'} + 1}{\dfrac{V}{V'\,s'} + \dfrac{1}{s'}}$$

$$= \frac{V\,s + V'\,s'}{V + V'}.$$

153. *In an alloy compounded of two known metals, to determine the proportion in which either of the metals enters.*

Let the alloy be compounded of the metals A and B whose specific gravities are s and s' respectively, and let s'' be the sp. gr. of the alloy. Let x be the weight of the quantity of the metal A contained in a mass of the alloy, whose weight is X. Then the weight of the metal B contained in the same mass is $X—x$. Hence Art. 151,

$$s'' = \frac{X}{\frac{x}{s} + \frac{X-x}{s'}}$$

$$\frac{x}{s} + \frac{X-x}{s'} = \frac{X}{s''}$$

$$\therefore \quad x\left(\frac{1}{s}-\frac{1}{s'}\right) = X\left(\frac{1}{s''}-\frac{1}{s'}\right)$$

$$\frac{x}{X} = \frac{\frac{1}{s''}-\frac{1}{s'}}{\frac{1}{s}-\frac{1}{s'}}$$

$$= \frac{s}{s''}\,\frac{s'-s''}{s'-s}.$$

Or if it be required to determine the proportion with respect to volume. Let y be the volume of the metal A contained in a volume Y of the alloy. Then $Y—y$ is the volume of the metal B contained in the alloy. Therefore, Art. 152,

$$s'' = \frac{y\,s + (Y-y)\,s'}{Y}$$

$$y\,(s-s') = Y\,(s''-s')$$

$$\therefore \quad \frac{y}{Y} = \frac{s''-s'}{s-s}$$

$$= \frac{s'-s''}{s'-s}.$$

154. *To determine the specific gravity of a fluid by means of the specific gravity bottle.*

The specific gravity bottle is simply a small flask fitted with a ground stopper.

Let w be the weight of the flask, x its weight when filled with the given fluid, and y its weight when filled with distilled water. Then $x—w =$ the weight of the given fluid contained in the flask, and $y—w =$ the weight of a similar volume of distilled water.

$$\therefore \qquad \text{sp. gr. of the fluid} = \frac{x-w}{y-w}.$$

Specific gravity bottles are very frequently made so as to contain exactly 1000 grains of pure distilled water. Then if x and w are known in grains.

$$\text{sp. gr. of the fluid} = \frac{x-w}{1000}.$$

155. *To determine the specific gravity of a fluid by weighing a solid body in it.*

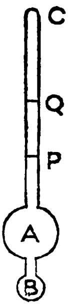
Let w be the weight of the solid body, x its weight when suspended in the given fluid, and y its weight when suspended in distilled water. Then, Art. 144, $w—x$ is the weight of a quantity of the given fluid equal in bulk to the solid body, and $w—y$ is the weight of a similar volume of distilled water.

$$\therefore \qquad \text{sp. gr. of the given fluid} = \frac{w-x}{w-y}.$$

Or if s be the specific gravity of the solid, then since (Art. 149), $s = \dfrac{w}{w-y}$.

$$\text{sp. gr. of the given fluid} = s.\frac{w-x}{w}.$$

156. *To determine the specific gravity of a fluid by means of the common hydrometer.*



The Common Hydrometer consists of a small hollow sphere A, to which there is attached, on one side, a slender graduated stem, and on the other a smaller sphere B, made of such a weight as to allow the whole instrument to float, with the sphere A entirely immersed.

When placed in distilled water, let the instrument sink as far as P, and when placed in the fluid whose specific gravity is sought, let it sink to Q.

Let W be the weight of the Hydrometer, and V its volume. Let a be the area of the section of the stem. Let $C P = x$, and $C Q = y$.

By Art. 145, when a body floats, the weight of the body is equal to that of the fluid displaced.

The quantity of water displaced is equal to the volume of the hydrometer, minus the part $C P$, that is, is equal to $V—a x$. Therefore,

$W =$ the weight of a quantity of water, whose volume is $V—a x$.

$$\therefore \quad \frac{W}{V—a x} = \text{weight of a cubic unit of water.}$$

Also: $W =$ the weight of a quantity of the given fluid whose volume is $V—a y$.

$$\therefore \quad \frac{W}{V—a y} = \text{weight of a cubic unit of the given fluid.}$$

And $\therefore$ $$\text{sp. gr. of the given fluid} = \frac{W}{V-a\,y} : \frac{W}{V-a\,x} = \frac{V-a\,x}{V-a\,y}.$$

157. The absolute weight of any substance may be easily found from a table of specific gravities.

A cubic foot of pure water at a temperature of 60°, weighs 1000 ounces avoirdupois.

If the specific gravity of a substance A be s, it follows from the definition of specific gravity that,

$$\frac{\text{weight of a cubic foot of } A}{\text{weight of a cubic foot of water}} = s,$$

and therefore,

$$\text{weight of a cubic foot of } A = s \times 1000 \text{ ounces}.$$

Thus for example to find the weight of a cubical block of marble, whose side is 10 feet; the specific gravity of the marble being 2·7;

$$\begin{aligned}\text{weight of a cubic foot of marble} &= 2{\cdot}7 \times 1000 \text{ ounces}\\ &= 2700 \quad \text{,,}\\ &= 168{\cdot}75 \text{ lbs.}\end{aligned}$$

but the block contains 10^3, or 1000 cubic feet of marble. Therefore,

$$\begin{aligned}\text{weight of the block} &= 168{\cdot}75 \times 1000 \text{ lbs.}\\ &= 168750 \text{ lbs.}\\ &= 75{\cdot}334 \text{ tons.}\end{aligned}$$

158. TABLE OF SPECIFIC GRAVITIES.

Platinum (forged)	21·4	Plate Glsss	2·9
Gold	19·3	Marble	2·7
Mercury (congealed) ...	15·6	Green Glass	2·6
„ (fluent)	13·6	Flint (black)	2 6
Lead	11·3	Portland Stone	2·5
Silver	10·5	Brick	2·0
Copper	8·9	Ivory	1·8
Steel	7·8	Lignum Vitæ	1·3
Iron (forged)	7·7	Ebony	1·2
Tin	7·3	Mahogany ·64 to	1·1
Iron cast at Carron ...	7·2	Beech	·85
Zinc compressed	7·2	Oak (English)	·75
„ in its common state	6·9	Scotch Fir	·69
Potassium	·97	Spruce Fir from Canada	·52
Sodium	·86	Cork	·24

CHAPTER IV.

ON ATMOSPHERIC PRESSURE.

159. The atmosphere surrounding the earth, being a fluid acted on by gravity, presses upon all bodies immersed in it, in accordance with the general laws of fluid pressure.

The pressure of the atmosphere upon any body will consequently vary with the depth of that body below its highest surface. As, however, the dimensions of any ordinary body upon the surface of the earth are inconsiderable, when compared with the height of the atmosphere, this pressure may, without any sensible error, be regarded as the same at all points in any such body.

It is found by experiment that the pressure which the atmosphere exerts upon a body, on or near the surface of the earth, is the same as that exerted by a column of water 32 feet in height, or by a column of mercury 30 inches in height, that is, it presses with a weight of about 15lbs. upon every square inch of surface. That we are not, in ordinary circumstances, cognizant of this pressure arises from the characteristic property of fluids, in accordance with which they press in all directions upon bodies immersed in them; so that, if the upper surface of a body subject to the pressure of the atmosphere, experience a downward pressure of 15lbs. to every square inch, the under surface experiences a corresponding upward pressure. If, however, we destroy this equilibrium, by removing or diminishing the pressure of the atmosphere upon one side of any body, we are immediately made sensible of the pressure which is exerted upon the opposite side. For instance, if a plate of glass be placed upon the top of a cylindrical receiver connected with an air pump, and the air be exhausted from the receiver, the plate of glass will be pressed down with a considerable force, and if not strong enough will be broken in pieces.

160. *When the atmosphere presses upon any point of a fluid mass, to determine the amount of transmitted pressure.*

If the atmosphere presses upon the highest surface of the fluid mass, then it is evident that the transmitted pressure is equal to the pressure of the atmosphere.

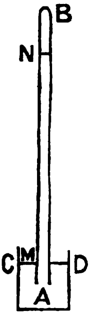
If the atmosphere presses at a point in the fluid mass below the highest surface, then a portion of the atmospheric pressure will be employed in counterbalancing the outward pressure of the fluid itself, and the remaining portion only of the atmospheric pressure will be transmitted by the fluid. Let h be the distance below the highest surface of the fluid mass of the point at which the atmosphere presses upon it, and let w be the weight of a cubic unit of the fluid, then as seen in Chapter II, the fluid itself will exert at this point an outward pressure on every square unit $= w\,h$. Then if Π denote the pressure exerted by the atmosphere upon a square unit, the transmitted pressure will be

$$\Pi - w\,h$$

for every unit of surface in the fluid mass.

If the point at which the atmosphere presses upon the fluid mass be so taken, that $w\,h$ is greater than Π, then it is plain that equilibrium cannot subsist, but motion will ensue amongst the particles of the fluid, until the distance becomes such, that $w\,h = \Pi$.

161. *The Barometer.* This instrument is constructed in the following manner;—a glass tube closed at one end is filled with mercury, and a finger being placed over the open end, is inverted in an open vessel of mercury.



Let $A\,B$ be such a tube, let $C\,D$ be the surface of the mercury in the vessel, and let $M\,B$ be more than 30 inches. Then since $C\,D$ is more than 30 inches below B the highest surface of the fluid mass, the pressure of the fluid upon any square unit in the surface $C\,D$ will be greater than the atmospheric pressure, Art. *159.* The mercury will consequently fall in the tube. Let N be point at which it rests, then if Π be the atmospheric pressure, and w the weight of a cubic unit of mercury

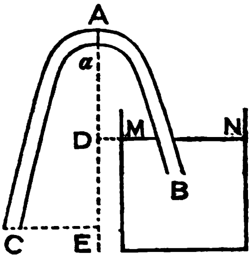
$$\Pi = w \times M\,N.$$

The height of the mercury in the tube will accordingly vary as the atmospheric pressure varies; will rise as the atmospheric pressure increases, and fall as it decreases.

A graduated scale is attached to the tube $A\,B$, by means of which the changes in the height of the mercury in the tube can be observed and measured.

162. *The Siphon.* The Siphon is a bent tube, open at both ends, and having one leg shorter than the other. If such a tube be filled with any fluid, and placed with the

extremity of its shorter leg in a vessel containing the same fluid, the fluid will flow out through the longer leg.



Let *B A C* be the siphon filled with fluid, and having *B* the extremity of its shorter leg *A B*, placed below *M N* the surface of the fluid in the vessel. Let *A* be the highest point of the siphon, and through *A* draw the vertical line *A E*. Let *M N* produced meet this vertical line in *D*, and let *C E* be the horizontal line drawn through *C*.

Let *A a* be a thin vertical film of the fluid passing through *A*. Since each side of this film is similarly situated with respect to *A* the highest point of the fluid mass, the pressure arising from the weight of the fluid will be the same on each side. These pressures, therefore, are in equilibrium, and do not cause the motion of the fluid.

But if $\Pi =$ the atmospheric pressure upon a square unit, and $w =$ the weight of a cubic unit of the fluid, then, since the atmosphere presses upon the surface *M N* at a distance *A D* below the highest point of the fluid mass, there is transmitted up the leg *B A*, by Art. 160, a pressure upon every square unit, equal to

$$\Pi - w \times A D.$$

Similarly, since the atmosphere presses upon the surface of the fluid mass at C, there is transmitted up $C\,A$ a pressure upon every square unit equal to

$$\Pi - w \times A\,E.$$

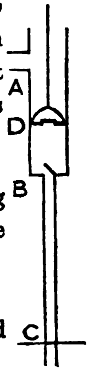
Hence the one side of the film $A\,a$ will experience a pressure upon every square unit equal to $\Pi - w \times A\,D$, and the other side a pressure equal to $\Pi - w \times A\,E$. But since $A\,D$ is less than $A\,E$, $\Pi - w \times A\,D$ is greater than $\Pi - w \times A\,E$; the film of fluid will therefore be forced down the leg $A\,C$. The same will happen to every film that may in succession occupy the position $A\,a$, the fluid consequently will flow out at C, so long as there remains in the vessel any fluid above B.

It has been assumed in the preceding, that the height $A\,D$ is such, that $w \times A\,D$ is less than Π. If $A\,D$ be so great that $w \times A\,D$ is greater than Π, then, as seen in the preceding articles, the fluid will sink in the leg $A\,B$ to some level below the bend, and the siphon consequently will cease to act.

163. *The Common Pump.* The Common Pump consists of a cylindrical barrel $A\,B$, furnished with a valve at B opening upwards, and with an air-tight piston capable of being moved up and down by means of a rod.

The piston is also furnished with a valve opening upwards, and to the barrel at B is attached a pipe $B\,C$, called the suction pipe.

Let C be the surface of the water to be raised, and suppose the piston to be at B.



Let the piston be raised from B to A; the air in $B\ C$ will, by virtue of its expansive force, press open the valve at B and fill the barrel $A\ B$, and in consequence of occupying a larger space $A\ B\ C$ will exert a less pressure than before upon the surface of the water within the pipe. Hence the pressure of the air within the pump being less than the atmospheric pressure upon the surface of the water, the water will be forced up the pipe $B\ C$ until equilibrium be restored.

Let the piston now descend,—the valve at B will be closed, and the air in $A\ B$ will escape through the valve in D. If the piston be again raised, the preceding process will be repeated, and the water will again rise in the pipe $B\ C$. In like manner upon every successive stroke of the piston the water will continue to rise in the pipe until, if $B\ C$ be less than 32 feet, it at last reaches B.

Let the water be at B, and let the piston be pressed down to B. Then since the atmosphere presses upon the surface of the fluid at C, there will be transmitted by the fluid a pressure upon every square unit of the valve at B equal to $\Pi - w \times B\ C$, Art. 160; and, consequently, when the piston is again raised, the water will force open the valve B, and will rise in the barrel as far as A, if $A\ C$ be less than 32 feet, or to some point between B and A, if $A\ C$ be greater than 32 feet.

Upon the next descent of the piston, the valve in D will be forced open, and the water will flow through it, so that when the piston reaches B the water in the barrel, will lie above the piston, and when the piston is raised will be raised along with it, and flow out at the spout A.

*164. *In a common pump, whose spout is less than* 32 *feet from the surface of the water, to determine the resultant* *pressure upon the piston at any moment during its ascent after it has commenced discharging the water.*

Let the piston be at D, then, since the water has commenced to be discharged, the barrel and pipe are filled with water, and since the piston is rising the valve at B will be open, and that at D will be shut.

The pressure upon the upper side of the piston is the sum of the weight of the fluid mass in $A\,D$, and the atmospheric pressure upon its surface. Therefore, upon every square unit of the upper surface of the piston, there is exerted a downward pressure equal to

$$\Pi + w \times A\,D.$$

Upon every square unit of the lower surface of the piston, there is, by Art. 160, an upward pressure equal to

$$\Pi - w \times D\,C.$$

Let a = the number of square units in the area of the piston, let $A\,C = h$, and $A\,D = x$. Then $D\,C$ will equal $h - x$. Therefore,

the total pressure upon the upper surface $= a\,\Pi + w\,a\,x$

" " lower " $= a\Pi - w\,a(h - x)$

The resultant pressure will be the difference of these pressures, and therefore equals

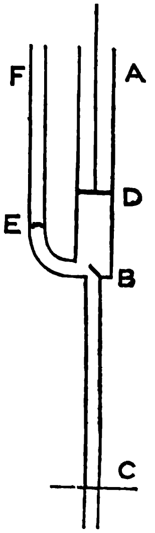
$$w\,a\,h.$$

Therefore the resultant pressure is independent of the position of D; in other words, the pressure upon the piston is the same at every point in its ascent after commencing to discharge, and is equal to the weight of a column of water, whose base equals the area of the piston, and whose height is the height of the spout of the pump above the surface of the water.

165. *The Forcing Pump.* The construction of the forcing pump is represented in the accompanying figure.

D is a piston with no valve, working in in a barrel *A B*; *B C* is a pipe descending to the water, *E F* a tube leading upwards from *B*, at *B* and *E* are valves opening upwards.

If the piston be raised from *B* to *A* then as in the common pump, the air in *B C* will press open the valve at *B* and fill the barrel, and the water will rise to a certain height in *B C*. As the piston descends the air in the barrel will be forced out through the valve at *E*; after a certain number of ascents the water will, if *B C* be less than 32 feet, pass into the barrel *A B*; and upon its next descent the piston will force the water in the barrel through the valve *E* into the pipe *E F*. As the piston continues to rise more water will pass into the barrel, and this in its turn will be forced by the descent of the piston into the pipe *E F*.



*166. *To determine the resultant pressure upon the piston of a forcing pump, at any moment during its ascent or descent, after it has commenced discharging.*

First.—Let the piston be ascending; the valve at *B* will be open, and that at *E* will be shut. Then if Π be the atmospheric pressure upon a square unit, w the weight of a cubic unit of water, and a the area of the piston, the downward pressure upon the upper surface of the piston will be

$$a\,\Pi,$$

and the upward pressure upon the lower surface of the piston will be

$$a\,(\Pi - w \times D\,C)$$

$$= a\,\Pi - w\,a \times D\,C.$$

Therefore, the resultant pressure will act downwards and be equal to

$$w\,a \times D\,C.$$

Hence, since as the piston rises, $D\,C$ increases, the resultant pressure increases at every moment of the ascent, and at any instant is equal to the weight of a column of water, whose base equals the area of the piston, and whose height is the distance of the piston from the surface of the water A.

Secondly.—Let the piston be descending, the valve B will be closed, and that at E will be open Let F be the highest point in the pipe $E\,F$.

The pressure upon the upper surface of the piston, will as before be, a downward force equal to

$$a\,\Pi.$$

The pressure upon the lower surface of the piston will be an upward force equal to

$$a\,\Pi + w\,a \times D\,F.$$

The resultant pressure will therefore be an upward force equal to

$$w\,a \times D\,F.$$

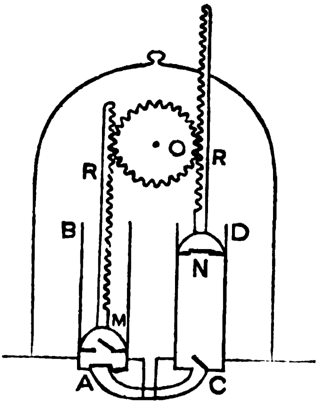
Hence, the pressure upon the piston increases at every moment of the descent, and at any instant equals the weight of a column of water, whose base equals the area of the piston, and whose height is the distance of the piston below the point at which the pipe discharges.

CHAPTER V.

ON THE AIR PUMP AND STEAM ENGINE.

167. *The double barrel Air Pump.*

A B and *C D* are two cylindrical barrels connected together by the pipe *A C*, and communicating by means of another pipe with an air tight receiver *R*. *M* and *N* are pistons furnished with valves opening upwards, and are worked by the toothed wheel *O*, so that when *M* descends *N* will ascend, and *vice versa*. At *A* and *C* are valves opening upwards.



Let *M* be at *A*, and *N* at *D*, and the wheel be turned so that *N* may descend and *M* ascend. As *N* descends the valve at *C* will close, and the air in the barrel *C D* will pass out through the valve in *N*. As *M* ascends, the pressure of the external air will close the valve in *M*; and the pressure being removed from the upper side of the valve A, the air in the receiver will in consequence of its expansive force press open the valve and fill the barrel $A\ B$. The air originally in the receiver now occupying a larger space, viz., the receiver and barrel together, will be of diminished density.

If the wheel be again turned a similar process will take place, and the air in the receiver will again expand, so as to fill both receiver and barrel, and a second diminution of density will consequently take place.

This process may be continued so long as the air in the receiver can by its expansive force press open the valves at A or C; but since the expansive force of the air is diminished with the diminution of its density, the action of the pump must after a certain number of strokes entirely cease.

*168. *To find the density of the air in the receiver after any number of turns of the wheel.*

If a quantity of air expand so as to occupy a space twice as great as before, its density will evidently be one half its original density; if the space be three times as great, the density will be one-third the original density; and so generally, the density after expansion will be to the density before expansion, as to the space occupied before expansion is to the space occupied after expansion.

Hence, if A be the content of the receiver, and B of each of the barrels, since at every turn the air in the receiver expands, and fills both receiver and barrel.

$$\frac{\text{density of the air at the close of any turn}}{\text{density of the air at the commencement of the turn}} = \frac{A}{A+B}$$

Consequently,

$$\frac{\text{density of the air after 1 turn}}{\text{density of atmospheric air}} = \frac{A}{A+B}$$

also,

$$\frac{\text{density of the air after 2 turns}}{\text{density of the air after 1 turn}} = \frac{A}{A+B}$$

and ∴

$$\frac{\text{density of the air after 2 turns}}{\text{density of atmospheric air}} = \left(\frac{A}{A+B}\right)^2.$$

In like manner,

$$\frac{\text{density of the air after 3 turns}}{\text{density of the air after 2 turns}} = \frac{A}{A+B}$$

and ∴

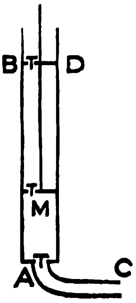
$$\frac{\text{density of the air after 3 turns}}{\text{density of atmospheric air}} = \left(\frac{A}{A+B}\right)^3.$$

Hence generally,

$$\frac{\text{density of the air after } n \text{ turns}}{\text{density of atmospheric air}} = \left(\frac{A}{A+B}\right)^n.$$

This result shews that, even if there were no practical limits to the working of an air pump, the air could never be entirely exhausted, for the density of the air in the receiver always has some assignable ratio to the density of atmospheric air.

169. *The Single-Barrel Air Pump.*



This Air Pump consists of a single barrel AB communicating with an air tight receiver by the pipe C.

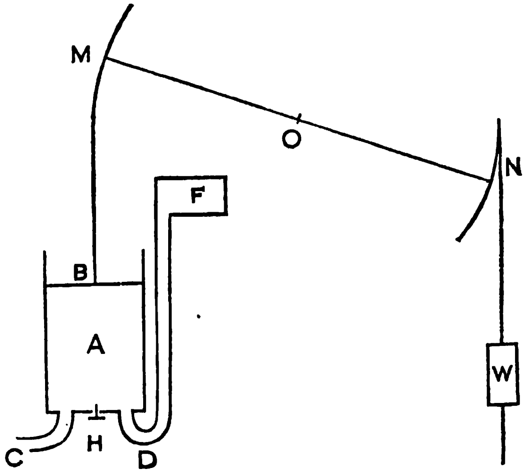
The piston passes through an air tight collar in the plate $B\,D$. At B is a valve opening upwards.

The process of exhaustion is precisely similar to that described in Art. 167.

The effect of the valve at *B* is to relieve the piston from the pressure of the atmosphere, during a part of its ascent. For if the air in *B M* be of less density than atmospheric air, the pressure of the atmosphere will press down the valve at *B*, and will keep it closed until the piston by its ascent has so compressed the air in *B M*, that its density has become greater than that of atmospheric air.

The plate *B D* with its valve opening upwards, is not peculiar to the single barrel air pump, but may, if required, be adopted in a double-barrel air pump. In fact it is used in the more carefully constructed double-barrel pumps.

170. *The Atmospheric Steam Engine.*



B is a solid piston working in the hollow cylinder *A*, and connected with one extremity of the beam *M N* by a chain attached to the piston-rod. *C* is a pipe leading from the boiler, and *D* a pipe leading from a reservoir of cold water *E*; *C* and *D* are both furnished with stop cocks. *H* is a valve opening downwards.

The beam *M N* moves about its centre *O*, and to the extremity *N* is attached a weight or counterpoise *W*, equal to half the atmospheric pressure upon the surface of the piston *B*.

Suppose the piston *B* to be at the bottom of the cylinder, and let the pressure exerted by the steam in the boiler be equal to that exerted by the atmosphere. If the cock in *C* be now opened, the steam will enter the cylinder, and will press upon the under surface of the piston with a force equal to the atmospheric pressure upon the upper surface. The weight *W* will consequently descend, and by its descent will raise the piston *B* to the top of the cylinder.

Let the cock in *C* be now closed, and that in *D* be opened; a jet of cold water will issue into the cylinder, and condensing the steam in *A* will leave a vacuum below *B*. Then, since the pressure of the atmosphere upon *B* is double the weight *W*, *B* will be pressed down with a force equal to *W*.

When *B* arrives at the bottom of the cylinder, the cock in *D* is closed, and that in *C* is opened, and the piston ascends as before.

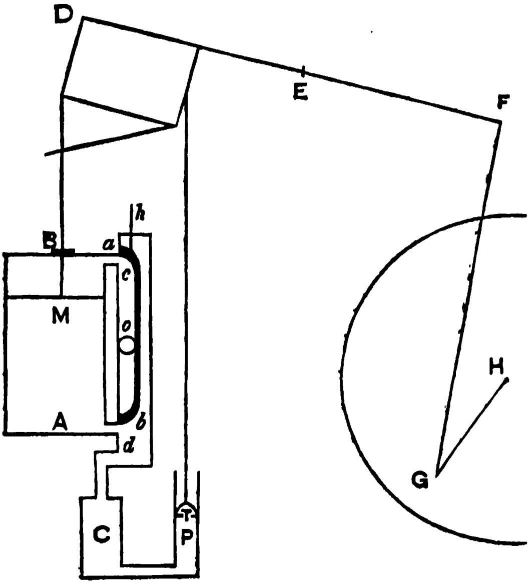
The water in the cylinder escapes by the valve *H*, which is forced open by the pressure of the steam when first admitted, and is conveyed by a pipe into a cistern communicating with the boiler.

*171. *The single acting Steam Engine.* This engine differs from that described in the preceding article in employing instead of atmospheric pressure, the pressure of steam to impel the piston downwards. It is thus in the strictest sense a *steam* engine.

The top of the cylinder is closed, with the exception of a passage for the piston-rod. This passage is furnished with a stuffing box filled with tow saturated with grease, so that while the piston-rod can move freely up and down, the steam is prevented from escaping. The steam is admitted into the cylinder above the piston, at the moment that a vacuum has been produced below the piston by condensation ; the pressure of the steam then forces the piston downwards. When the piston has arrived at the lowest point, the steam which fills the cylinder above the piston is allowed to circulate below the piston, (a communication between the upper and lower parts of the cylinder being opened for this purpose). The piston is then pressed equally by the steam upon its upper and lower surfaces, and will consequently be raised by the weight attached to the beam.

In the single acting Steam Engine the pressure of the steam—as in the atmospheric engine the pressure of the atmosphere—acts only during the descent of the piston ; and in both engines the force necessary to raise the counterpoise, was so much subtracted from the available power of the engine. These objections were obviated by the invention of the double acting steam engine, in which the upward as well as the downward stroke was caused by the pressure of the steam.

*172. *The Double-acting Steam Engine.*



A B is a hollow cylinder closed at both ends. *M* is a solid piston. The piston rod passes through a stuffing box at *B*, and is connected with one extremity of the beam *D E F*; the other extremity of the beam is attached to the crank of a fly wheel. *C* is a vessel, called the condenser, surrounded with cold water, *a d* is a box communicating with the upper and lower parts of the cylinder *A B*, with the condenser *C*, and also, by means of a steam pipe passing from an orifice at *o*, with the boiler.

A moveable slide *a b* works in the box *a d*, and regulates the passage of the steam from the boiler into the cylinder, and from the cylinder into the condenser. When the slide is in the position represented in the figure, the steam from the boiler can pass into the upper part only of the cylinder, and the steam below the piston can escape into the condenser. If the slide be pressed down, by means of the rod *h*, so that *a* shall come to *c* and *b* to *d*, the steam from the boiler can pass into the lower part only of the cylinder, and the steam above the piston can escape into the condenser.

Suppose *M* to be at the top of the cylinder, and the space below *M* to be filled with steam from .the boiler. Let the slide be moved into the position represented into the figure. The steam below the piston will pass into *C*, and being there condensed will leave a vacuum in *A M*. At the same time the steam from the boiler will pass into the cylinder above the piston, and the pressure of the steam will force the piston down.

Let the slide be now moved into its lower position. The steam above the piston will flow into the condenser, and leave a vacuum in *B M*. At the same time the steam from the boiler will enter the cylinder below the piston, and the piston will again be raised.

The water formed in *C* by the condensation of the steam, is removed by a pump *P*.

*173. The engines described in the preceding articles may be arranged under the general class of *condensing steam engines*. In engines of this class, the condensation of steam is employed to destroy or to diminish the pressure upon one side of the piston. Other engines are termed *non-condensing*.

*174. *The non-condensing Steam Engine.* In this engine the condenser and the somewhat cumbrous apparatus connected therewith are not required. The structure of the engine is in consequence considerably simplified, and its weight greatly diminished. And for these reasons Locomotive engines are always non-condensing.

The non-condensing steam engine is worked by steam of a pressure greater than the atmospheric pressure. Let steam of such a pressure enter the upper part of the cylinder, and at the same time let a communication be opened between the lower part of the cylinder and the atmosphere. The pressure of the steam on the upper surface of the piston being greater than the atmospheric pressure on the lower, the piston will descend.

Let the steam from the boiler now enter the lower part of the cylinder, and let a communication be opened between the upper part of the cylinder and the atmosphere. The steam in the upper part of the cylinder will escape into the atmosphere, and the pressure upon the lower surface of the piston being greater than that upon its upper surface, the piston will ascend.

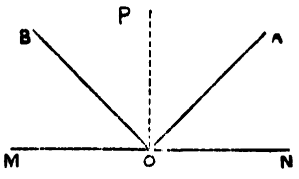
Non-condensing engines are popularly, but improperly, called *high-pressure Steam Engines*, and in contradistinction, the condensing engines are called *low-pressure Steam Engines*.

APPENDIX.

ON THE LAWS OF LIGHT.

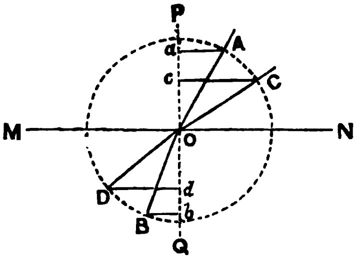
When a ray of light falls upon any surface, the angle which it makes with the perpendicular to the surface at the point of incidence, is termed the *angle of incidence;* the angle which the reflected ray makes with the same perpendicular is termed the *angle of reflection,* and similarly the angle which the refracted ray makes with this perpendicular is termed the *angle of refraction.*

Law of Reflection. The reflected ray lies in the same plane with the incident ray and the perpendicular to the surface at the point of incidence; and the angle of reflection is equal to the angle of incidence.



Thus if $A\ O$ be a ray of light incident upon the reflecting surface $M\ N$, and if $O\ B$ be the reflected ray, and $O\ P$ the perpendicular to the surface, then will $O\ B$ lie in the same plane with $A\ O$ and $O\ P$, and the angle $B\ O\ P$ be equal to the angle $A\ O\ P$.

Law of Refraction. The refracted ray lies in the same plane with the incident ray and the perpendicular to the surface at the point of incidence; and the angles of incidence and refraction are related in the following manner;



Let $A\ O$ be a ray incident upon $M\ N$ the surface of a refracting medium. Let $O\ B$ be the refracted ray, and $P\ Q$ the perpendicular to the surface drawn through O. With O as a centre, and any distance as a radius, describe a circle cutting the incident, and refracted rays in A and B. From A and B draw $A\ a$, $B\ b$ perpendicular to $P\ Q$, and let m represent the ratio of $A\ a$ to $B\ b$, that is, let

$$m = \frac{A\ a}{B\ b}$$

Then if $C\ O$ represent any other ray incident npon the surface $M N$ at O, and $O D$ be the direction of this ray after refraction, and if $C\ c$, $D\ d$ be the perpendiculars to $P\ Q$, the ratio of $C\ c$ to $D\ d$ will also be m, that is, the refracted ray will make with $P\ Q$ such an angle that,

$$\frac{C\ c}{D\ d} = m.$$

The quantity m is termed the *refractive index* and the law of refraction affirms, that for any two given media the value of m remains the same, however the angle of incidence may vary.

When light passes from a rarer into a denser medium, the value of m is greater than unity, and consequently the angle of refraction is less than the angle of incidence. When light passes from a denser into a rarer medium m is less than unity, and the angle of refraction greater than the angle of incidence. Hence, by refraction, a ray of light is bent *towards* the perpendicular when passing into a denser medium, and *from* the perpendicular when passing into a rarer medium.

Between atmospheric air, and plate or crown glass, $m = \frac{3}{2}$ nearly

ON THE FORMATION OF IMAGES BY LENSES.

Any portion of a refracting medium which has two opposite surfaces, either both spherical, or one spherical and the other plane is termed a *lens*.

The *axis* of a lens is the line joining the centres of the two surfaces, when both are spherical, but when one of the surfaces is a plane, is the perpendicular to the plane through the centre of the spherical surface.

Lenses are of two classes, convex and concave. *Convex* lenses are those which are thickest at the centre. *Concave* lenses those which are thinnest at the centre.

When rays of light diverging from a luminous point fall upon a lens, the refracted rays will be less divergent than the incident rays of the lens be convex, but more divergent if the lens be concave. A pencil of divergent rays *may* therefore become convergent after refraction by a convex lens, but cannot become so after refraction by a concave lens.

If the refracted rays either converge to, or diverge from a single point, that point is termed the *focus* of the refracted rays. In the former case the focus is *real*, and the rays meeting there will form a luminous spot; in the latter case the focus is *virtual*.

When rays of light diverging from a luminous point fall upon any spherical lens, the refracted rays do not in any case proceed exactly to or from a single point. Those rays, however, which are near the axis, will, after refraction, proceed to or from a certain point so nearly, that their distance from it is not appreciable; this point is therefore regarded as a focus.

A luminous body may be regarded as a collection of luminous points. If such a body be situated in the axis of a lens, from each point rays will fall upon the surface of the lens, and some of these rays will in each case after refraction converge to a focus or diverge from one. For each point in the given body, there will therefore be a new point from which rays of light will proceed. This collection of points constitutes what is termed the IMAGE of the given body. When the refracted rays converge to a focus, the image is *real*, but when they diverge from a focus the image is *virtual*. When the image is real, if the refracted rays be received upon a screen placed in the focus, the image or picture of the original object will be made visible, and appear as if painted on the screen.

ON THE NATURE OF SOUND.

When the particles of an elastic body are suddenly displaced at any point, a peculiar motion termed *vibration* is communicated to all the particles of that body. The displaced particle oscillates or vibrates through small distances about its position of equilibrium, its excursions on either side gradually diminishing, and at length altogether ceasing. A similar vibratory movement is communicated to the other particles of the body. The velocity with which a vibration is transmitted from one point in the body to another, is different in different bodies, and must carefully be distinguished from the absolute velocity of the particles themselves.

An elastic body in a state of vibration, may communicate a similar motion to the particles of another elastic body in contact with it.

The sensation of sound is produced by the vibration of an elastic body in contact with the ear.

If air be the medium, in contact with the ear, as is most commonly the case, sound is produced by the vibrations of the air; if water be the medium in contact with the ear, the vibrations of the water produce the sensation, if an iron or glass rod, then the vibrations of the particles of the rod produce it.

The vibratory motion which produces the sensation of sound is either produced directly in the medium in contact with the ear, or is communicated to that medium by the vibration of another elastic body. When a bell is struck, a vibratory motion is given to the particles of the bell, and we hear the sound, because vibrations are thence communicated to the surrounding air, which is the medium in contact with our ear. If, however, the bell be so placed, that its vibrations cannot be communicated to the air, no sound will be heard, however violently it may be rung. For instance, if the bell be suspended under the receiver of an air pump, and be made to ring, as rarefaction proceeds the sound becomes gradually weakened, and at length becomes inaudible.

The velocity of Sound. The average velocity of the propagation of sound in atmospheric air is found by experiment to be about 1130 feet in a second. In water the velocity of sound is 4900 feet in a second, and along a deal rod the velocity is as great as 17,000 feet in a second.

Increase of temperature increases the velocity of sound. According to the best experiments, the velocity of sound in air at 62° Fahrenheit, is 1125 feet in a second. Every increase or decrease of temperature of 1° Fahrenheit, causes a corresponding increase or decrease of 1·14 feet in the velocity of sound. Accordingly the velocity of sound in air at the freezing point will be 1090 feet per second.

In any given medium, the velocity of propagation is found to be the same for all sounds, whatever their loudness or tone.

Musical Tones. If a succession of vibrations performed in different times strike upon the ear, the sound produced is called *noise*, but if a succession of vibrations performed in equal times strike upon the ear, an effect pleasing to the ear is produced, and the sound is distinguished as a *tone*. It is found, however, that to produce this effect, there must be at least 30 double vibrations in a second.

Tones differ according to the number of vibrations performed in a second. That sound is called the higher, which is caused by the greater number of vibrations. The note represented by the middle *A* of the treble clef, requires 420 double vibrations per second. The note *C* next below this, is the effect of 252 double vibrations per second.

THE END.

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